

# A SKEW STOCHASTIC HEAT EQUATION

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**ABSTRACT.** We consider a stochastic heat equation driven by a space-time white noise and with a singular drift, where a local-time in space appears. The process we study has an explicit invariant measure of Gibbs type, with a non-convex potential. We obtain existence of a Markov solution, which is associated with an explicit Dirichlet form. Moreover we study approximations of the stationary solution by means of a regularization of the singular drift or by a finite-dimensional projection.

## 1. INTRODUCTION

**1.1. The skew Brownian motion.** Consider the following stochastic differential equation in  $\mathbb{R}$ :

$$X_t = X_0 + B_t + \beta L_t^0, \quad t \geq 0, \quad (1.1)$$

where  $(B_t)_{t \geq 0}$  is a standard Brownian motion in  $\mathbb{R}$ ,  $(L_t^0)_{t \geq 0}$  is the local time at 0 of the process  $(X_t)_{t \geq 0}$ , namely

$$L_t^0 = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{(|X_s| \leq \varepsilon)} ds. \quad (1.2)$$

Harrison and Shepp [15] have proved that equation (1.1)-(1.2) has a unique solution iff  $|\beta| \leq 1$  and there is no solution if  $|\beta| > 1$ . In the former case, the process  $(X_t)_{t \geq 0}$  has the law of the *skew Brownian motion* with parameter  $\alpha = (1 + \beta)/2$ , i.e. a Brownian motion whose excursions are chosen to be positive, respectively negative, independently of each other, and each with probability  $\alpha$ , resp.  $1 - \alpha$ .

In this paper we want to introduce a stochastic heat equation which has some analogy with (1.1)-(1.2). Let us also note that an invariant measure for  $(X_t)_{t \geq 0}$  is given by

$$m_\alpha(dx) = (1 - \alpha)\mathbb{1}_{(x>0)} dx + \alpha\mathbb{1}_{(x<0)} dx = C \exp(-c \mathbb{1}_{(x>0)}(x)) dx,$$

where  $c, C$  are constants depending on  $\alpha$ . Moreover  $(X_t)_{t \geq 0}$  is associated with the Dirichlet form in  $L^2(m_\alpha)$

$$E(u, v) := \frac{1}{2} \int_{\mathbb{R}} u' v' dm_\alpha.$$

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2000 *Mathematics Subject Classification.* Primary: 60H07; 60H15; 60J55; Secondary 31C25.

*Key words and phrases.* Stochastic partial differential equations; Local time; Dirichlet Forms; Gamma convergence.

**1.2. A skew SPDE.** In this paper we want to study a *skew stochastic heat equation*, namely the stochastic partial differential equation (SPDE)

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\alpha}{2} \frac{\partial}{\partial \theta} \ell_\theta^0 + \dot{W}, \\ u(t, 0) = u(t, 1) = 0, \\ u(0, \theta) = u_0(\theta), \quad \theta \in [0, 1] \end{cases} \quad (1.3)$$

where  $(\ell_{t,\theta}^a, \theta \in [0, 1])$  is the family of local times at  $a \in \mathbb{R}$  accumulated over  $[0, \theta]$  by the process  $(u(t, r), r \in [0, 1])$ ,  $W(t, \theta)$  is a Brownian sheet over  $[0, +\infty[ \times [0, 1]$  and  $\dot{W}(t, \theta)$  is therefore a space-time white-noise and  $u_0 \in L^2(0, 1)$ . In fact, we consider a more general version of equation (1.3), see (1.6) below.

We recall that the *stochastic heat equation* is given by

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2} \frac{\partial^2 v}{\partial \theta^2} + \dot{W}, \\ v(t, 0) = v(t, 1) = 0 \\ v(0, \theta) = u_0(\theta), \quad x \in [0, 1] \end{cases} \quad (1.4)$$

The process  $(v_t, t \geq 0)$  is an-infinite dimensional Ornstein-Uhlenbeck process and it is associated with the Dirichlet form

$$\mathcal{E}^0(\varphi, \psi) := \frac{1}{2} \int_H \langle \nabla \varphi, \nabla \psi \rangle d\mu,$$

in  $L^2(\mu)$ , where  $H := L^2(0, 1)$ ,  $\nabla$  is the Fréchet gradient on  $H$  and  $\mu$  is the law of a standard Brownian bridge from 0 to 0 over  $[0, 1]$ , see [6].

Equation (1.3) is naturally associated with a perturbation of  $\mathcal{E}^0$ , defined by means of the probability measure on  $H$

$$\nu(dx) := \frac{1}{Z} \exp \left( -\alpha \int_0^1 \mathbb{1}_{(x_s > 0)} ds \right) \mu(dx),$$

with  $\alpha \in \mathbb{R}$ , and of the Dirichlet form

$$\mathcal{E}(\varphi, \psi) := \frac{1}{2} \int_H \langle \nabla \varphi, \nabla \psi \rangle d\nu, \quad (1.5)$$

in  $L^2(\nu)$ . Equation (1.3) is therefore a natural infinite-dimensional version of (1.1): indeed, its invariant measure  $\nu$  favors paths over  $[0, 1]$  which spend more time in the positive axis than in the negative one. The definition and construction of this process are non-trivial, for several reasons.

First, the local-time term plays the role of a very singular drift, which furthermore lacks any dissipativity property; this makes a well-posedness result difficult to expect. Secondly, the explicit invariant measure  $\nu$  is not *log-concave*, a condition which would insure a number of nice properties of the Dirichlet form  $\mathcal{E}$  and of the associated Markov process, see e.g. [2] and section 2.1 below.

In particular, the process is not Strong-Feller, or at least a proof of this property is out of our reach, see [5] for a host of examples and consequences of this nice continuity property. We are at least able to prove something weaker, namely the *absolute continuity* of the transition semigroup w.r.t. the invariant measure  $\nu$ , see Proposition 2.5 below; our proof of this technical step seems to be new and of independent interest.

We also consider two different regularizations of equation (1.6): first we approximate  $f$  with a sequence of smooth functions; then we consider finite-dimensional projections (without regularizing  $f$ ). In both cases we prove convergence in law of the associated stationary processes. The main technical tool is the  $\Gamma$ -convergence (or, in this context, the *Mosco-convergence*) of a sequence of Dirichlet forms with underlying Hilbert space depending on  $n$ . This notion has been introduced by Kuwae and Shioya in [17] as a generalization of the original idea of Mosco [19] and later developed by Kolesnikov in [16] for finite-dimensional and a particular class of infinite-dimensional problems. Our approach has been largely inspired by the recent work of Andres and von Renesse, see [3, 4].

**1.3. Main results.** We start by giving the main definition. We consider a bounded function  $f : \mathbb{R} \mapsto \mathbb{R}$  with bounded variation and we want to study the following equation

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial \theta^2} - \frac{1}{2} \int_{\mathbb{R}} f(da) \frac{\partial}{\partial \theta} \ell_{t,\theta}^a + \dot{W}, \\ u(t, 0) = u(t, 1) = 0, \\ u(0, \theta) = u_0(\theta), \quad \theta \in [0, 1] \end{cases} \quad (1.6)$$

where  $(\ell_{t,\theta}^a, \theta \in [0, 1])$  is the family of local times at  $a \in \mathbb{R}$  accumulated over  $[0, \theta]$  by the process  $(u(t, r), r \in [0, 1])$ .

**Definition 1.1.** Let  $x \in L^2(0, 1)$ . An adapted process  $u$ , defined on a complete filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ , is a weak solution of (1.6) if

- a.s.  $u \in C([0, T] \times [0, 1])$  and  $\mathbb{E}[\|u_t - x\|^2] \rightarrow 0$  as  $t \downarrow 0$
- a.s. for  $dt$ -a.e.  $t$  the process  $(u(t, r), r \in [0, 1])$  has a family of local times  $[0, 1] \times \mathbb{R} \ni (r, t) \mapsto \ell_{t,\theta}^a, a \in \mathbb{R}$ , such that

$$\int_0^\theta g(u(t, r)) dr = \int_{\mathbb{R}} g(a) \ell_{t,\theta}^a da, \quad \theta \in [0, 1], t \geq 0,$$

for all bounded Borel  $g : \mathbb{R} \mapsto \mathbb{R}$ .

- there is a Brownian sheet  $W$  such that for all  $h \in C_c^2((0, 1))$  and  $0 < \varepsilon \leq t$

$$\begin{aligned} \langle u_t - u_\varepsilon, h \rangle &= \frac{1}{2} \int_\varepsilon^t \langle h'', u_s \rangle_{L^2(0,1)} ds + \frac{1}{2} \int_\varepsilon^t \int_{\mathbb{R}} f(da) \int_0^1 h'(\theta) \ell_{s,\theta}^a d\theta ds \\ &\quad + \int_\varepsilon^t \int_0^1 h(\theta) W(ds, d\theta) \end{aligned} \quad (1.7)$$

A Brownian sheet is a Gaussian process  $W = \{W(t, \theta) : (t, \theta) \in \mathbb{R}_+^2\}$  defined on  $(\Omega, \mathcal{F}, \mathbb{P})$ , such that  $\{W(t, \theta) : \theta \in \mathbb{R}_+\}$  is  $\mathcal{F}_t$ -measurable for all  $t \geq 0$ , with zero mean and covariance function

$$\mathbb{E}[W(t, \theta)W(t', \theta')] = (t \wedge t')(\theta \wedge \theta'), \quad t, \theta, t', \theta' \in \mathbb{R}_+.$$

In section 2 we study the Dirichlet form  $\mathcal{E}$  defined by (1.5), proving in particular that it satisfies the absolute continuity condition, namely the resolvent operators have kernels which admit a density with respect to the reference measure  $\nu$ . In section 3 we show that the Markov process associated with  $\mathcal{E}$  is a weak solution of (1.6). Although for general  $f$  a uniqueness result for solutions to (1.6) seems to be out of reach, the process we construct is somewhat canonical, since it is associated with the Dirichlet form  $\mathcal{E}$  and moreover it is obtained as the limit of natural regularization/discretization procedures, as shown in sections 4, respectively 5. Indeed, in section 4 we regularize the nonlinearity  $f$  and show that the (stationary) solutions to the approximated equations converge to the stationary solution of (1.6). In section 5 we show convergence of finite-dimensional processes, obtained via a space-discretization, to the solution of (1.6).

**1.4. Motivations.** There is an extensive literature on reaction-diffusion stochastic partial differential equations of the form

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial \theta^2} - \frac{1}{2} f'(u) + \dot{W}, \quad t \geq 0, \theta \in [0, 1],$$

see for instance the monography by Cerrai [5]; note that by the occupation times formula, for smooth  $f$  this equation is equivalent to (1.6). This kind of equation has also been used as a model for fluctuations of effective interface models, see [13]. However, in order to give a sense to the above equation, it is typically assumed that  $f$  is smooth or convex. In this paper we study this equation in the case where  $f$  is neither convex nor necessarily smooth and can even have jumps.

One of the motivations of this work is given by the problem of extending the results of [14] on convergence of fluctuations of a stochastic interface near a hard wall to a non log-concave situation. In particular, it is a long standing problem to prove the same result as in [14] for a critical pinning model, see e.g. [9], where the invariant measure converges in the limit to the law of a reflecting Brownian motion. Such a situation is highly non log-convex and the techniques developed for instance in [2] do not apply. In this paper we show that the  $\Gamma$ -convergence is an effective tool also in this context.

**1.5. Notations.** We consider the Hilbert space  $H = L^2(0, 1)$  endowed with the canonical scalar product

$$\langle h, k \rangle_H := \int_0^1 h_\theta k_\theta d\theta, \quad \|h\|^2 := \langle h, h \rangle, \quad h, k \in H.$$

$$C_0 := C_0(0, 1) := \{c : [0, 1] \mapsto \mathbb{R} \text{ continuous, } c(0) = c(1) = 0\},$$

$$A : D(A) \subset H \mapsto H, \quad D(A) := W^{2,2} \cap W_0^{1,2}(0, 1), \quad A := \frac{1}{2} \frac{d^2}{d\theta^2}.$$

We introduce the following function spaces:

- We denote by  $C_b(H)$  the space of all  $\varphi : H \mapsto \mathbb{R}$  being bounded and uniformly continuous in the norm of  $H$ . We let  $\|\varphi\|_\infty := \sup |\varphi|$ . Then  $(C_b(H), \|\cdot\|_\infty)$  is a Banach space.
- We denote by  $\text{Exp}_A(H)$  the linear span of  $\{1, \cos(\langle \cdot, h \rangle), \sin(\langle \cdot, h \rangle) : h \in D(A)\}$ .
- The space  $\text{Lip}(H)$  is the set of all  $\varphi \in C_b(H)$  such that:

$$\|\varphi\|_{\text{Lip}} := \|\varphi\|_\infty + \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\|x - y\|} < \infty.$$

- The space  $C_b^1(H)$  is defined as the set of all Fréchet-differentiable  $\varphi \in C_b(H)$ , with continuous bounded gradient  $\nabla \varphi : H \mapsto H$ .

We sometimes write:  $m(\varphi)$  for  $\int_H \varphi dm$ ,  $\varphi \in C_b(H)$ .

## 2. THE DIRICHLET FORM $\mathcal{E}$

In this section we give a detailed study of the Dirichlet form  $\mathcal{E}$ , proving in particular that it satisfies the *absolute continuity property*, see Proposition 2.5 below.

**2.1. A non-log-concave probability measure.** Let  $\beta = (\beta_\theta, \theta \in [0, 1])$  be a standard Brownian bridge and let us denote its law by  $\mu$ . Then  $\mu$  is a Gaussian measure on the Hilbert space  $H = L^2(0, 1)$ . We consider a bounded function  $f : \mathbb{R} \mapsto \mathbb{R}$  with bounded variation and we define  $F : H \mapsto \mathbb{R}$ :

$$F(x) := \int_0^1 f(x_r) dr, \quad x \in H.$$

We define the probability measure on  $H$

$$\nu(dx) = \frac{1}{Z} \exp(-F(x)) \mu(dx), \quad Z := \int \exp(-F) d\mu. \quad (2.1)$$

where  $Z$  is normalizing constant. Note that  $f$  is not assumed to be convex, and therefore  $\nu$  is in general not log-concave, see [2]. Finally we have clearly

$$\frac{1}{C} \|\cdot\|_{L^2(\mu)}^2 \leq \|\cdot\|_{L^2(\nu)}^2 \leq C \|\cdot\|_{L^2(\mu)}^2 \quad (2.2)$$

for some constant  $C > 0$ , since  $f$  is bounded.

**2.2. The Gaussian Dirichlet Form.** We define now

$$\mathcal{E}^0(\varphi, \psi) := \frac{1}{2} \int_H \langle \nabla \varphi, \nabla \psi \rangle d\mu, \quad \forall \varphi, \psi \in C_b^1(H).$$

Then it is well known that the symmetric positive bilinear form  $(\mathcal{E}^0, \text{Exp}_A(H))$  is closable in  $L^2(\mu)$ , see e.g. [7]: we denote by  $(\mathcal{E}^0, D(\mathcal{E}^0))$  the closure. We recall that  $\mu$ , law of a standard Brownian bridge  $\beta$ , has covariance  $Q := (-2A)^{-1}$ , a compact operator on  $H$  which can be diagonalized as follows:

$$Qh = \sum_{k=1}^{\infty} \lambda_k \langle h, e_k \rangle_H e_k, \quad h \in H,$$

where

$$\lambda_k := \frac{1}{(\pi k)^2}, \quad e_k(x) := \sqrt{2} \sin(k\pi x), \quad x \in [0, 1], \quad k \in \mathbb{N}^*.$$

It is well known that the Markov process defined by (1.4), i.e. the solution of the stochastic heat equation, is associated with the Dirichlet form  $(\mathcal{E}^0, D(\mathcal{E}^0))$  in  $L^2(\mu)$ . This process is Gaussian and can be written down explicitly as a stochastic convolution. We recall the following result from [7]:

**Proposition 2.1.** *Let  $\Gamma := \{\gamma : \mathbb{N}^* \mapsto \mathbb{N} : \sum_k \gamma_k < +\infty\}$ . Then there exists a complete orthonormal system  $(H_\gamma)_{\gamma \in \Gamma}$  in  $L^2(\mu)$  such that*

$$\mathcal{E}^0(\varphi, \varphi) = \sum_{\gamma \in \Gamma} \Lambda_\gamma \langle \varphi, H_\gamma \rangle_{L^2(\mu)}^2, \quad \forall \varphi \in D(\mathcal{E}^0),$$

where  $\Lambda_\gamma$  is given by

$$\Lambda_\gamma := \sum_{k \in \mathbb{N}^*} \gamma_k \lambda_k^{-1}. \quad (2.3)$$

In particular, the embedding  $D(\mathcal{E}^0) \mapsto L^2(\mu)$  is compact.

It follows that  $(H_\gamma)_{\gamma \in \Gamma}$  is a c.o.s. of eigenvalues of the Ornstein-Uhlenbeck operator associated with  $\mathcal{E}^0$ . We denote by  $(P_t^0)_{t \geq 0}$  the associated semigroup in  $L^2(\mu)$ , which can be of course written as

$$P_t^0 \varphi = \sum_{\gamma \in \Gamma} e^{-\Lambda_\gamma t} \langle \varphi, H_\gamma \rangle_{L^2(\mu)} H_\gamma, \quad \forall \varphi \in L^2(\mu).$$

Then we have the following

**Proposition 2.2.** *For all  $t > 0$  the operator  $P_t^0 : L^2(\mu) \mapsto L^2(\mu)$  is Hilbert-Schmidt, i.e.*

$$\sum_{\gamma \in \Gamma} e^{-2\Lambda_\gamma t} = \prod_{k=1}^{\infty} \frac{1}{1 - e^{-2t\pi^2 k^2}} < +\infty, \quad t > 0. \quad (2.4)$$

In particular, the series

$$p_t^0(x, y) := \sum_{\gamma \in \Gamma} e^{-\Lambda_\gamma t} H_\gamma(x) H_\gamma(y)$$

converges in  $L^2(\mu \otimes \mu)$  and yields an integral representation of  $P_t^0$ :

$$P_t^0 \varphi(x) = \int \varphi(y) p_t^0(x, y) \mu(dy), \quad \mu\text{-a.e. } x, \quad \forall \varphi \in L^2(\mu).$$

*Proof.* Let us define  $C_n$ , for  $n \in \mathbb{N}$ , as the number of  $\gamma \in \Gamma$  such that  $\sum_k \gamma_k k^2 = n$ . Then

$$\sum_{\gamma \in \Gamma} e^{-2\Lambda_\gamma t} = \sum_{\gamma \in \Gamma} \sum_{n=0}^{\infty} \mathbb{1}_{(\Lambda_\gamma=n)} e^{-2\Lambda_\gamma t} = \sum_{n=0}^{\infty} C_n e^{-2\pi^2 t n}.$$

Now, by a classical formula due to Euler, the generating function of the sequence  $(C_n)_{n \geq 0}$  is given by

$$\chi(r) := \sum_{n=0}^{\infty} C_n r^n = \prod_{k=1}^{\infty} \frac{1}{1 - r^{k^2}}, \quad |r| < 1.$$

The infinite product converges, since by taking the logarithm

$$-\log(1 - r^{k^2}) \sim r^{k^2}, \quad k \rightarrow +\infty, \quad |r| < 1,$$

which is a summable sequence. By choosing  $r = e^{-2t\pi^2}$ , the first claim follows. The rest is a trivial consequence of this result.  $\square$

From (2.4) one can obtain the following

**Proposition 2.3.** *The embedding  $D(\mathcal{E}^0) \mapsto L^2(\mu)$  is not Hilbert-Schmidt.*

*Proof.* The embedding  $D(\mathcal{E}^0) \mapsto L^2(\mu)$  is Hilbert-Schmidt if and only if

$$\sum_{\gamma \in \Gamma \setminus \{0\}} \frac{1}{\Lambda_\gamma} < +\infty.$$

Again we can write

$$\sum_{\gamma \in \Gamma \setminus \{0\}} \frac{1}{\Lambda_\gamma} = \sum_{\gamma \in \Gamma} \sum_{n=1}^{\infty} \mathbb{1}_{(\Lambda_\gamma=n)} \frac{1}{\Lambda_\gamma} = \sum_{n=1}^{\infty} \frac{C_n}{n}.$$

Now, using the generating function  $\chi$  of the sequence  $C_n$  we obtain

$$\sum_{n=1}^{\infty} \frac{C_n}{n} = \int_0^1 dr \sum_{n=1}^{\infty} C_n r^{n-1} = \int_0^1 \frac{\chi(r) - 1}{r} dr,$$

since  $C_0 = 1$ . The latter integral converges near 0, but it diverges near 1, since  $\chi(r) \geq (1-r)^{-1}$ . Therefore the above sum is infinite.  $\square$

**2.3. The Dirichlet form associated with (1.6).** We define the symmetric positive bilinear form

$$\mathcal{E}(\varphi, \psi) := \frac{1}{2} \int_H \langle \nabla \varphi, \nabla \psi \rangle d\nu, \quad \forall \varphi, \psi \in C_b^1(H).$$

Let us set  $\mathcal{K} := \text{Exp}_A(H)$ .

**Lemma 2.4.** *The symmetric positive bilinear form  $(\mathcal{E}, \mathcal{K})$  is closable in  $L^2(\nu)$ . We denote by  $(\mathcal{E}, D(\mathcal{E}))$  the closure.*

*Proof.* By (2.2) we have that

$$\frac{1}{C} \mathcal{E}_1^0 \leq \mathcal{E}_1 \leq C \mathcal{E}_1^0. \quad (2.5)$$

Closability of  $(\mathcal{E}^0, \mathcal{K})$  yields immediately the result.  $\square$

**2.4. Absolute continuity.** Let  $(P_t)_{t \geq 0}$  be the semigroup associated with the Dirichlet form  $(\mathcal{E}, D(\mathcal{E}))$  in  $L^2(\nu)$ . We denote by  $R_\lambda := \int_0^\infty e^{-\lambda t} P_t dt$ ,  $\lambda > 0$ , the resolvent family of  $(P_t)_{t \geq 0}$ . In this section we want to prove the following

**Proposition 2.5.** *There exists a measurable kernel  $(\rho_\lambda(x, dy), \lambda > 0, x \in H)$  such that*

$$R_\lambda \varphi(x) = \int \varphi(y) \rho_\lambda(x, dy), \quad \nu\text{-a.e. } x, \quad \forall \varphi \in L^2(\nu),$$

and such that for all  $\lambda > 0$  and for all  $x \in H$  we have  $\rho_\lambda(x, dy) \ll \nu(dy)$ .

We are going to use the following result, see [10, pp. 1543].

**Theorem 2.6** (Minimax principle). *Let  $(T, \mathcal{D}(T))$  a self-adjoint linear operator on the separable Hilbert space  $\mathbb{H}$  such that  $T \geq 0$  and  $(\lambda - T)^{-1}$  is a compact operator for some  $\lambda > 0$ . We denote by  $\mathcal{S}^n$  the family of  $n$ -dimensional subspace of  $\mathbb{H}$ , and for  $n \geq 1$  we let  $\lambda_n$  the number defined as follows*

$$\lambda_n := \sup_{G \in \mathcal{S}^n} \inf_{u \in (G \cap \mathcal{D}(T)) \setminus \{0\}} \frac{\langle u, Tu \rangle_{\mathbb{H}}}{\langle u, u \rangle_{\mathbb{H}}}. \quad (2.6)$$

Then there exists a complete orthonormal system  $(\psi_n)_{n \geq 1}$  such that

$$T \psi_n = \lambda_n \psi_n, \quad n \geq 1.$$

In other words, the sequence  $(\lambda_n)_{n \geq 1}$  is the non-decreasing enumeration of the eigenvalues of  $T$ , each repeated a number of times equal to its multiplicity. Moreover the sup in (2.6) is attained for  $G$  equal to the span of  $\{\psi_1, \dots, \psi_n\}$ .

With the help of Theorem 2.6, we can first prove the following

**Proposition 2.7.** *The operator  $P_t : L^2(\nu) \mapsto L^2(\nu)$  is Hilbert-Schmidt and there exists a function  $p_t \in L^2(\nu \otimes \nu)$  such that*

$$P_t \varphi(x) = \int \varphi(y) p_t(x, y) \nu(dy), \quad \nu\text{-a.e. } x, \quad \forall \varphi \in L^2(\nu).$$

*Proof.* We recall that an analogous result has been proved in Proposition 2.2 for the semigroup  $(P_t^0)_{t \geq 0}$  associated with the Dirichlet form  $(\mathcal{E}^0, D(\mathcal{E}^0))$  in  $L^2(\mu)$ . Now we want to deduce the same result for  $(P_t)_{t \geq 0}$ .

We apply first Theorem 2.6 to the Ornstein-Uhlenbeck operator  $L^0$  associated with  $(\mathcal{E}^0, D(\mathcal{E}^0))$  in  $L^2(\mu)$ . Since  $R_1^0 := (1 - L^0)^{-1}$  maps  $L^2(\mu)$  into  $D(\mathcal{E}^0)$  and the embedding  $D(\mathcal{E}^0) \hookrightarrow L^2(\mu)$  is compact by Proposition 2.3, then  $R_1^0$  is compact and also symmetric since  $\mathcal{E}^0$  is symmetric. By Proposition 2.3, the spectrum of  $(-L^0)$  is pure point, its eigenvalues are  $(\Lambda_\gamma)_{\gamma \in \Gamma}$  and the associated eigenvectors are the c.o.s.  $(H_\gamma)_{\gamma \in \Gamma}$ . If we call  $(\delta_n^0)_{n \geq 1}$  the non-decreasing enumeration of  $(\Lambda_\gamma)_{\gamma \in \Gamma}$ , then by the above result we obtain that

$$\delta_n^0 := \sup_{G \in \mathcal{S}^n} \inf_{u \in (G \cap D(L^0)) \setminus \{0\}} \frac{\mathcal{E}^0(u, u)}{\langle u, u \rangle_{L^2(\mu)}}.$$



In fact, since the sup above is attained for  $G$  equal to the span of  $\{\psi_1, \dots, \psi_n\} \subseteq D(\mathcal{E}^0)$ , then we can also write

$$\delta_n^0 = \sup_{G \in \mathcal{S}^n} \inf_{u \in (G \cap D(\mathcal{E}^0)) \setminus \{0\}} \frac{\mathcal{E}^0(u, u)}{\langle u, u \rangle_{L^2(\mu)}}.$$

In the same way, setting

$$\delta_n := \sup_{G \in \mathcal{S}^n} \inf_{u \in (G \cap D(\mathcal{E})) \setminus \{0\}} \frac{\mathcal{E}(u, u)}{\langle u, u \rangle_{L^2(\nu)}},$$

then  $(\delta_n)_{n \geq 1}$  is the non-decreasing enumeration of the eigenvalues of  $(-L) : D(L) \subset L^2(\nu) \mapsto L^2(\nu)$ . Now, by (2.2) and (2.5), we obtain that

$$\frac{1}{C} \delta_n^0 \leq \delta_n \leq C \delta_n^0, \quad n \geq 1.$$

Therefore for  $t > 0$

$$\sum_n e^{-2t\delta_n} \leq \sum_n e^{-2t\frac{1}{C}\delta_n^0}$$

and the latter sum is finite by (2.4). Therefore  $P_t : L^2(\nu) \mapsto L^2(\nu)$  is Hilbert-Schmidt, symmetric and non-negative. Then Proposition 2.7 follows from a well-known characterization of operators with such properties.  $\square$

*Proof of Proposition 2.5.* In [12, Theorem 7.2.1] it is proved that there exist a set of zero capacity  $N$  and a measurable Markov kernel  $(p_t(x, dy), t \geq 0, x \in N^c)$  on  $N^c$ , such that the function  $x \mapsto \int \varphi(y) p_t(x, dy)$  is  $\nu$ -a.s. equal to  $P_t \varphi$  and quasi-continuous on  $N^c$  for all  $t, > 0$ . By quasi-continuity we want to say that there is a sequence of nondecreasing closed set  $(F_n)_n$ , with no isolated point, such that the previous map, restricted on  $F_n$ , is continuous for all  $t > 0$  and  $N^c = \cup_n F_n$ . By Proposition 2.7, for  $\nu$ -a.e.  $x$  we have  $p_t(x, dy) = p_t(x, y) \nu(dy)$ , with  $p_t \in L^2(\nu \otimes \nu)$  and  $p_t \geq 0$ ,  $\nu \otimes \nu$ -almost surely. It follows that the kernel  $\rho_\lambda(x, dy)$  representing the resolvent operator  $R_\lambda := \int_0^\infty e^{-\lambda t} P_t dt$  is in fact given for  $\nu$ -a.e.  $x$  by  $\rho_\lambda(x, dy) = \rho_\lambda(x, y) \nu(dy)$ , where for  $\nu \otimes \nu$ -a.e.  $(x, y)$

$$\rho_\lambda(x, y) := \int_0^{+\infty} e^{-\lambda t} p_t(x, y) dt.$$

Moreover  $R_\lambda \varphi$  is continuous on  $N^c$  for all  $\varphi \in L^2(\nu)$ . This allows to prove that  $\rho_\lambda(x, dy) \ll \nu(dy)$  for all  $x \in N$ : indeed, if  $B$  is a measurable set such that  $\nu(B) = 0$ , then  $\rho_\lambda(x, B) = 0$  for  $\nu$ -a.e.  $x$  and therefore, by density and continuity, for all  $x \in N^c$ . As in [12], we can set  $\rho_\lambda(x, dy) \equiv 0$  for all  $x \in N$ , and the proof is complete.  $\square$

### 3. EXISTENCE OF A SOLUTION

In this section we want to prove the following

**Proposition 3.1.** *The Dirichlet form  $(\mathcal{E}, D(\mathcal{E}))$  is quasi-regular and the associated Markov process is a weak solution of equation (1.6).*

We recall here the basics of potential theory which are needed in what follows, referring to [12] and [18] for all proofs. By Proposition 3.1, the Dirichlet form  $(\mathcal{E}, D(\mathcal{E}))$  is *quasi-regular*, i.e. by [18, Theorem IV.5.1] it can be embedded into a regular Dirichlet form; in particular, the classical theory of [12] can be applied. Moreover, the important *absolute continuity condition* of Proposition 2.5 allows to pass from the stationary solution to quasi-every initial condition: see for instance [12, Theorem 4.1.2 and formula (4.2.9)].

We denote by  $\mathcal{F}_\infty^\lambda$  (resp.  $\mathcal{F}_t^\lambda$ ) the completion of  $\mathcal{F}_\infty^0$  (resp. completion of  $\mathcal{F}_t^0$  in  $\mathcal{F}_\infty^\lambda$ ) with respect to  $\mathbb{P}_\lambda$  and we set  $\mathcal{F}_\infty := \cap_{\lambda \in \mathcal{P}(K)} \mathcal{F}_\infty^\lambda$ ,  $\mathcal{F}_t := \cap_{\lambda \in \mathcal{P}(K)} \mathcal{F}_t^\lambda$ , where  $\mathcal{P}(K)$  is the set of all Borel probability measures on  $K$ .

*Capacity and Additive functionals.* Let  $A$  be an open subset of  $H$ , we define by  $\mathcal{L}_A := \{u \in D(\mathcal{E}) : u \geq 1, \nu\text{-a.e. on } A\}$ . Then we set

$$\text{Cap}(A) = \begin{cases} \inf_{u \in \mathcal{L}_A} \mathcal{E}_1(u, u), & \mathcal{L}_A \neq \emptyset, \\ +\infty & \mathcal{L}_A = \emptyset, \end{cases}$$

where  $\mathcal{E}_1$  is the inner product on  $D(\mathcal{E})$  defines as follow

$$\mathcal{E}_1(u, v) = \mathcal{E}(u, v) + \int_H u(x) v(x) d\nu, \quad u, v \in D(\mathcal{E}).$$

For any set  $A \subset H$  we let

$$\text{Cap}(A) = \inf_{B \text{ open}, A \subset B \subset H} \text{Cap}(B)$$

A set  $N \subset H$  is *exceptional* if  $\text{Cap}(N) = 0$ .

By a Continuous Additive Functional (CAF) of  $X$ , we mean a family of functions  $A_t : E \mapsto \mathbb{R}^+, t \geq 0$ , such that:

- (A.1)  $(A_t)_{t \geq 0}$  is  $(\mathcal{F}_t)_{t \geq 0}$ -adapted
- (A.2) There exists a set  $\Lambda \in \mathcal{F}_\infty$  and a set  $N \subset H$  with  $\text{Cap}(N) = 0$  such that  $\mathbb{P}_x(\Lambda) = 1$  for all  $x \in H \setminus N$ ,  $\theta_t(\Lambda) \subseteq \Lambda$  for all  $t \geq 0$ , and for all  $\omega \in \Lambda$ :  $t \mapsto A_t(\omega)$  is continuous,  $A_0(\omega) = 0$  and for all  $t, s \geq 0$ :

$$A_{t+s}(\omega) = A_s(\omega) + A_t(\theta_s \omega),$$

where  $(\theta_s)_{s \geq 0}$  is the time-translation semigroup on  $E$ .

Moreover, by a Positive Continuous Additive Functional (PCAF) of  $X$  we mean a CAF of  $X$  such that:

- (A.3) For all  $\omega \in \Lambda$ :  $t \mapsto A_t(\omega)$  is non-decreasing.

Two CAFs  $A^1$  and  $A^2$  are said to be equivalent if

$$\mathbb{P}_x(A_t^1 = A_t^2) = 1, \quad \forall t > 0, \forall x \in K \setminus N.$$

If  $A$  is a linear combination of PCAFs of  $X$ , the Revuz measure of  $A$  is a Borel signed measure  $\Sigma$  on  $K$  such that:

$$\int_H \varphi d\Sigma = \int_H \mathbb{E}_x \left[ \int_0^1 \varphi(X_t) dA_t \right] \nu(dx), \quad \forall \varphi \in C_b(H).$$

From theorem VI.2.4 of [18], the correspondence between the PCAF and its Revuz measure is one-to-one

*The Fukushima decomposition.* Let  $h \in C_0^2((0, 1); \mathbb{R}^d)$ , and set  $U : H \mapsto \mathbb{R}$ ,  $U(x) := \langle x, h \rangle$ . By Theorem 3.1, the Dirichlet Form  $(\mathcal{E}, D(\mathcal{E}))$  is quasi-regular. Therefore we can apply the Fukushima decomposition, as it is stated in Theorem VI.2.5 in [18], p. 180: for any  $U \in \text{Lip}(H) \subset D(\mathcal{E})$ , we have that there exist an exceptional set  $N$ , a Martingale Additive Functional of finite energy  $M^{[U]}$  and a Continuous Additive Functional of zero energy  $N^{[U]}$ , such that for all  $x \in K \setminus N$ :

$$U(X_t) - U(X_0) = M_t^{[U]} + N_t^{[U]}, \quad t \geq 0, \mathbb{P}_x - \text{a.s.} \quad (3.1)$$

*Smooth measures.* We recall now the notion of smoothness for a positive Borel measure  $\Sigma$  on  $H$ , see [12, page 80]. A positive Borel measure  $\Sigma$  is *smooth* if

- (1)  $\Sigma$  charges no set of zero capacity
- (2) there exists an increasing sequence of closed sets  $\{F_n\}$  such that  $\Sigma(F_n) < \infty$ , for all  $n$  and  $\lim_{n \rightarrow \infty} \text{Cap}(K - F_n) = 0$  for all compact set  $K$ .

By definition, a signed measure  $\Sigma$  on  $H$  is smooth if its total variation measure  $|\Sigma|$  is smooth. That happens if and only if  $\Sigma = \Sigma^1 - \Sigma^2$ , where  $\Sigma^1$  and  $\Sigma^2$  are positive smooth measures, obtained from  $\Sigma$  by applying the Jordan decomposition, see [12, page 221].

We recall a definition from [12, Section 2.2]. We say that a positive Radon measure  $\Sigma$  on  $H$  is *of finite energy* if for some constant  $C > 0$

$$\int |v| d\Sigma \leq C \sqrt{\mathcal{E}_1(v, v)}, \quad \forall v \in D(\mathcal{E}) \cap C_b(H). \quad (3.2)$$

If (3.2) holds, then there exists an element  $U_1 \Sigma$  such that

$$\mathcal{E}_1(U_1 \Sigma, v) = \int_H v d\Sigma, \quad \forall v \in D(\mathcal{E}) \cap C_b(H).$$

Moreover, by [12, Lemma 2.2.3], all measures of finite energy are smooth.

Finally, by [12, Theorem 5.1.4], if  $\Sigma$  is a positive smooth measure, then there exists a PCAF  $(A_t)_{t \geq 0}$ , unique up to equivalence, with Revuz measure equal to  $\Sigma$ .

### 3.1. The associated Markov process.

We have first the following

**Lemma 3.2.** *The Dirichlet form  $(\mathcal{E}, D(\mathcal{E}))$  is quasi-regular.*

*Proof.* By (2.5) and by [18, Definition IV.3.1], quasi-regularity of  $(\mathcal{E}, D(\mathcal{D}))$  follows from quasi-regularity of  $(\mathcal{E}^0, D(\mathcal{D}^0))$ , which in turns follows from the fact that this Dirichlet form is associated with the solution to the stochastic heat equation (1.4).  $\square$

By [18, Theorem IV.3.5], quasi-regularity implies existence of a Markov process associated with  $(\mathcal{E}, D(\mathcal{E}))$ .

*Existence of local times.*

**Proposition 3.3.** *Almost surely, for a.e.  $t$  there exists a bi-continuous family of local times  $[0, 1] \ni (r, a) \mapsto \ell_{t,r}^a$  of  $(u_t(\theta), \theta \in [0, 1])$ .*

*Proof.* Let us recall that  $\nu$  is equivalent to the law  $\mu$  of  $(\beta_r, r \in [0, 1])$ , where  $\beta$  is a Brownian bridge over  $[0, 1]$ . Since  $\beta$  is a semi-martingale, for  $\mu$ -a.e.  $x$  there exists a family of local times  $\ell_r^a$  such that

$$\int_0^r g(x_s) ds = \int_{\mathbb{R}} g(a) \ell_r^a da, \quad r \in [0, 1],$$

and the map  $[0, 1] \times \mathbb{R} \ni (r, a) \mapsto \ell_r^a \in \mathbb{R}$  is continuous. In particular, setting

$$S := \{w \in C([0, 1]) : w \text{ has a bi-continuous family of local times } (\ell_r^a)_{(r,a) \in [0,1] \times \mathbb{R}}\},$$

then  $\nu(S) = 1$  and therefore

$$\mathbb{E}_x \left[ \int_0^t \mathbb{1}_{(u_s \in S^c)} ds \right] = \int_0^t \mathbb{P}_x(u_s \in S^c) ds = \int_0^t p_s(x, S^c) ds = 0$$

since the law of  $(u_s(\theta), \theta \in [0, 1])$  by Proposition 2.7 is absolutely continuous w.r.t.  $\nu$ . Therefore, the time spent by  $(u_s, s \geq 0)$  in  $S^c$  is a.s. equal to 0.  $\square$

We need now an integration by parts formula on the Dirichlet form  $\mathcal{E}$ . We recall the definitions

$$F(x) := \int_0^1 f(x_r) dr, \quad \rho(x) := \exp(-F(x)), \quad x \in H,$$

where  $f : \mathbb{R} \mapsto \mathbb{R}$  is a bounded function with bounded variation.

**Proposition 3.4.** *For any  $h \in D(A)$  and  $\varphi \in C_b^1(H)$*

$$\mathbb{E}[\rho(\beta) \partial_h \varphi(\beta)] = \mathbb{E} \left[ \rho(\beta) \varphi(\beta) \left( -\langle h'', \beta \rangle + \int_{\mathbb{R} \times [0,1]} f(da) h_r \ell^a(dr) \right) \right]. \quad (3.3)$$

*Proof.* Let  $h \in D(A)$  and  $\varepsilon \in \mathbb{R}$ , by the occupation time formula:

$$\begin{aligned} F(\beta + \varepsilon h) &= \int_0^1 f(\beta_r + \varepsilon h_r) dr = \int_{\mathbb{R}} \int_0^1 f(a + \varepsilon h_r) \ell^a(dr) da \\ &= \int_{\mathbb{R} \times \mathbb{R} \times [0,1]} da f(da) \ell^a(dr) \mathbb{1}_{(a \geq s - \varepsilon h_r)} \quad \text{a.s.} \end{aligned}$$

where  $(\ell^a(r), a \in \mathbb{R}, r \in [0, 1])$  is the local times family of  $\beta$ . Therefore

$$\left. \frac{d}{d\varepsilon} F(\beta + \varepsilon h) \right|_{\varepsilon=0} = - \int_{\mathbb{R} \times [0,1]} f(da) h_r \ell^a(dr).$$

Then by using the Cameron-Martin formula

$$\mathbb{E}[\rho(\beta) \varphi(\beta + \varepsilon h)] = \mathbb{E}[\rho(\beta - \varepsilon h) \varphi(\beta) \exp(-\varepsilon \langle h'', \beta \rangle - \|h\|^2 \varepsilon^2 / 2)]$$

and by differentiating w.r.t.  $\varepsilon$  at  $\varepsilon = 0$  we conclude.  $\square$

We want now to show that the process associated with  $\mathcal{E}$  satisfies (1.6). We are going to apply (3.1) to  $U^h(x) := \langle x, h \rangle$ ,  $x \in H$ , with  $h \in C_c^2((0, 1); \mathbb{R}^d)$ . Clearly  $U^h \in \text{Lip}(H) \subset D(\mathcal{E})$ . Our aim is to prove the following

**Proposition 3.5.** *There is an exceptional set  $N$  such that for all  $x \in H \setminus N$ ,  $\mathbb{P}_x$ -a.s. for all  $t \geq 0$*

$$N_t^{[U^h]} = \frac{1}{2} \int_0^t \langle h'', u_s \rangle ds + \frac{1}{2} \int_{[0, t] \times [0, 1]} \int_{\mathbb{R}} f(da) h'_r \ell_{s,r}^a ds dr \quad (3.4)$$

where a.s. for all  $s > 0$

$$- \int_{[0, 1]} \int_{\mathbb{R}} h'_r \varphi(a) \ell_{s,r}^a dr = \int_0^1 h_r \varphi(u_s(r)) dr, \quad \forall \varphi \in C_b(\mathbb{R}).$$

*Proof.* The main tools of the proof are the integration by parts formula (3.3) and a number of results from the theory of Dirichlet forms in [12]. We start by applying (3.1) to  $U^h(x) := \langle x, h \rangle$ ,  $x \in H$ . By approximation and linearity we can assume that  $h \in D(A)$ ,  $h'' \geq 0$  and therefore  $h \geq 0$  as well. The process  $N^{[U^h]}$  is a CAF of  $X$ , and its Revuz measure is  $\frac{1}{2} \Sigma^h$ , where

$$\Sigma^h(dw) := \left( \langle w, h'' \rangle - \int_{\mathbb{R} \times [0, 1]} f(da) h_r d\ell_r^a \right) \nu(dw) \quad (3.5)$$

and  $\ell_r^a$  is the bi-continuous family of local times of the Brownian bridge. Remark that we have the estimate

$$\mathbb{E} \left( \left( \int_{\mathbb{R} \times [0, 1]} f(da) h_r d\ell_r^a \right)^2 \right) < +\infty$$

since  $f(da)$  has globally bounded variation,  $h$  is bounded and  $\ell_1^a$  is in  $L^p$  for any  $p \geq 1$ .

By linearity, it is enough to consider the case  $h \geq 0$ . Then the measurable function  $\Phi(w) := \int_{[0, 1] \times \mathbb{R}} h_r d\ell_r^a f(da)$  is non-negative, and  $\Phi d\nu$  is a measure with finite energy, since

$$\int |v| \Phi d\nu \leq \|\Phi\|_{L^2(\nu)} \|v\|_{L^2(\nu)} \leq \|\Phi\|_{L^2(\nu)} \sqrt{\mathcal{E}_1(v, v)}, \quad \forall v \in D(\mathcal{E}) \cap C_b(H),$$

see (3.2) above. In particular,  $\Phi d\nu$  is a smooth measure. By theorem 5.1.3 of [12], there is an associated PCAF, denoted by  $N_t$ . Notice that the process

$$N_t^n := \int_0^t (\Phi \wedge n)(X_s) ds$$

is a well defined PCAF with Revuz measure  $\Phi \wedge n d\nu$  and  $N_t^n \leq N_t$ , since  $N_t - N_t^n$  is a CAF with a non-negative Revuz measure. By monotone convergence we find

for all non-negative  $\varphi \in C_b(H)$

$$\begin{aligned} \int_H \varphi \Phi d\nu &= \lim_n \int_H \varphi \Phi \wedge n d\nu = \lim_n \mathbb{E}_\nu \left[ \int_0^1 \varphi(X_t) (\Phi \wedge n)(X_t) dt \right] \\ &= \mathbb{E}_\nu \left[ \int_0^1 \varphi(X_t) \Phi(X_t) dt \right]. \end{aligned}$$

Therefore,  $t \mapsto \int_0^t \Phi(X_s) ds$  is a PCAF with Revuz measure  $\Phi d\nu$  and must therefore be equivalent to  $t \mapsto N_t$ .  $\square$

**3.2. Identification of the noise term.** We deal now with the identification of  $M^{[U^h]}$  with the integral of  $h$  with respect to a space-time white noise.

**Proposition 3.6.** *There exists a Brownian sheet  $(W(t, \theta), t \geq 0, \theta \in [0, 1])$ , such that*

$$M_t^{[U^h]} = \int_0^t \int_0^1 h_\theta W(ds, d\theta), \quad h \in H. \quad (3.6)$$

*Proof.* We recall that, for  $U \in D(\mathcal{E})$ , the process  $M^{[U]}$  is a continuous martingale, whose quadratic variation  $(\langle M^{[U]} \rangle_t)_{t \geq 0}$  is a PCAF of  $X$  with Revuz measure  $\mu_{\langle M^{[U]} \rangle}$  given by the formula

$$\int f d\mu_{\langle M^{[U]} \rangle} = 2\mathcal{E}(Uf, U) - \mathcal{E}(U^2, f), \quad \forall f \in D(\mathcal{E}) \cap C_b(H), \quad (3.7)$$

see [12, Theorem 5.2.3]. Now, if we apply this formula to  $U^h(x) = \langle x, h \rangle$ , then we obtain

$$\int f d\mu_{\langle M^{[U^h]} \rangle} = \|h\|^2 \int f d\nu, \quad \forall f \in D(\mathcal{E}) \cap C_b(H).$$

Therefore, the quadratic variation  $\langle M^{[U^h]} \rangle_t$  is equal to  $\|h\|^2 t$  for all  $t \geq 0$ , and, by Lévy's Theorem,  $(M^{[U^h]} \cdot \|h\|^{-1})_{t \geq 0}$  is a Brownian motion. Moreover, the parallelogram law, if  $h_1, h_2 \in H$  and  $\langle h_1, h_2 \rangle = 0$ , then the quadratic covariation between  $M^{[U^{h_1}]}$  and  $M^{[U^{h_2}]}$  is equal to

$$\langle M^{[U^{h_1}]}, M^{[U^{h_2}]} \rangle_t = t \langle h_1, h_2 \rangle, \quad t \geq 0.$$

Therefore,  $(M_t^{[U^h]}, t \geq 0, h \in H)$  is a Gaussian process with covariance structure

$$\mathbb{E}_x \left( M_t^{[U^{h_1}]} M_s^{[U^{h_2}]} \right) = s \wedge t \langle h_1, h_2 \rangle.$$

If we define  $W(t, \theta) := M_t^{[U^h]}$  with  $h := 1_{[0, \theta]}$ ,  $t \geq 0$ ,  $\theta \in [0, 1]$ , then  $W$  is the desired Brownian sheet.  $\square$

*Proof of Proposition 3.1.* Quasi-regularity has been proved in Lemma 3.2. First we apply the Fukushima decomposition (3.1) to the function  $U_h(x) := \langle x, h \rangle$  and identify the terms using propositions 3.6 and 3.5 and the above results. It remains to prove that the process  $(X_t)_{t \geq 0}$  satisfies the desired continuity properties. To this

aim, we use the result of Lemma 6.1 below. We notice that for any  $\eta \in (0, 1/2)$  and  $p > 1$

$$\begin{aligned} \frac{1}{C} \int_H \|x\|_{W^{\eta,p}(0,1)}^p \nu(dx) &\leq \mathbb{E} \left( \|\beta\|_{W^{\eta,p}(0,1)}^p \right) \leq \mathbb{E} \left( |\beta_r|^p + \int_0^1 \int_0^1 \frac{|\beta_s - \beta_t|^p}{|s - t|^{p\eta+1}} dt ds \right) \\ &\leq 1 + \int_0^1 \int_0^1 |s - t|^{p(\frac{1}{2}-\eta)-1} dt ds < +\infty. \end{aligned}$$

Then by Lemma 6.1 and by Kolmogorov's criterion in the Polish space  $C^\beta([0, 1])$  we obtain that under  $\mathbb{P}_\nu$  the coordinate process has a modification in  $C([0, T] \times [0, 1])$  for all  $T > 0$ .

Finally, in order to prove continuity of a non-stationary solution, we use the absolute-continuity property of proposition 2.5. Let us consider the set  $C := C([0, 1])$  endowed with the uniform topology. Let  $S \subset ]0, +\infty[$  be countable and satisfying  $\varepsilon := \inf S > 0$  and  $\sup S < \infty$ , and define  $B_S \subset C^{[0, +\infty[}$  as

$$B_S := \left\{ \omega \in C^{[0, +\infty[} : \text{the restriction of } \omega \text{ to } S \text{ is uniformly continuous} \right\},$$

then we know that  $\mathbb{P}_\nu(B_S) = 1$ , i.e.  $\mathbb{P}_x(B_S) = 1$  for  $\nu$ -a.e.  $x$ . For all  $x \in N^c$ , where  $N$  is exceptional, the law of  $X_\varepsilon$  under  $\mathbb{P}_x$  is absolutely continuous w.r.t.  $\nu$  for all  $\varepsilon > 0$ . Then  $\mathbb{P}_{X_\varepsilon}(B_{S-\varepsilon}) = 1$ ,  $\mathbb{P}_x$ -almost surely. Taking expectations, and using the Markov property, we get  $\mathbb{P}_x(B_S) = 1$ . Arguing as in [21, Lemma 2.1.2] we obtain that  $\mathbb{P}_x^*(C([0, +\infty[; C)) = 1$ , where  $\mathbb{P}_\nu^*$  denotes the outer measure.  $\square$

#### 4. CONVERGENCE OF REGULARIZED EQUATIONS

In this section we consider a smooth approximation  $f_n$  of  $f$  and we study convergence in law of  $u^n$  to  $u$ , where

$$\begin{cases} \frac{\partial u^n}{\partial t} = \frac{1}{2} \frac{\partial^2 u^n}{\partial \theta^2} - \frac{1}{2} f'_n(u^n) + \dot{W}, \\ u^n(t, 0) = u^n(t, 1) = 0, \\ u^n(0, \theta) = u_0^n(\theta), \quad \theta \in [0, 1]. \end{cases} \quad (4.1)$$

By a  $\Gamma$ -convergence technique, we shall prove convergence in law of the stationary processes.

Since  $f$  is bounded and with bounded variation, then it is continuous outside a countable set  $\Delta_f$ . Moreover we can find a sequence of smooth functions  $f_n : \mathbb{R} \mapsto \mathbb{R}$  such that

- (1)  $(f_n)_n$  is uniformly bounded
- (2)  $f_n \rightarrow f$  as  $n \rightarrow +\infty$  locally uniformly in  $\mathbb{R} \setminus \Delta_f$ .

We define the probability measure on  $H$

$$\nu_n(dx) = \frac{1}{Z_n} \exp(-F_n(x)) \mu(dx), \quad Z_n := \int \exp(-F_n) d\mu, \quad (4.2)$$

where  $Z_n$  is a normalizing constant. Again,  $\nu_n$  is not necessarily log-concave, see [2]. Setting

$$\rho_0 := 1, \quad \rho_n := \frac{d\nu_n}{d\mu}, \quad n \geq 1, \quad \rho := \frac{d\nu}{d\mu},$$

we find that  $0 < c \leq \rho_n \leq C < +\infty$  and  $0 < c \leq \rho \leq C < +\infty$  on  $H$ , since  $f_n$  and  $f$  are bounded for all  $n \in \mathbb{N}$ . We have then the simple

**Lemma 4.1.** *There is a canonical identification between the Hilbert spaces  $L^2(\nu)$  and  $L^2(\nu_n)$  for all  $n \in \mathbb{N}$  and for positive constants  $c, C$*

$$\frac{c}{C} \|\cdot\|_{L^2(\nu)}^2 \leq \|\cdot\|_{L^2(\nu_n)}^2 \leq \frac{C}{c} \|\cdot\|_{L^2(\nu)}^2. \quad (4.3)$$

*Proof.* This is obvious since  $0 < c \leq \rho_n \leq C < +\infty$  and  $0 < c \leq \rho \leq C < +\infty$ .  $\square$

In particular we can consider  $L^2(\nu_n)$  as being a copy of  $L^2(\nu)$  endowed with a different norm  $\|\cdot\|_{L^2(\nu_n)}$ . We shall use this notation below.

We define the symmetric positive bilinear form

$$\mathcal{E}^n(\varphi, \psi) := \frac{1}{2} \int_H \langle \nabla \varphi, \nabla \psi \rangle d\nu_n, \quad \forall \varphi, \psi \in C_b^1(H),$$

Let us set  $\mathcal{K} := \text{Exp}_A(H)$ .

**Lemma 4.2.** *The symmetric positive bilinear forms  $(\mathcal{E}^n, \mathcal{K})$  is closable in  $L^2(\nu_n)$ . We denote by  $(\mathcal{E}^n, D(\mathcal{E}^n))$  the closure.*

*Proof.* The proof is identical to that of Lemma 2.4.  $\square$

We recall that the Dirichlet form  $(\mathcal{E}^n, D(\mathcal{E}^n))$  is associated with the solution of equation (4.1), see e.g. [7].

**4.1. Convergence of Hilbert spaces.** We recall now the following definition, given by Kuwae and Shioya in [17].

**Definition 4.3.** *A sequence of Hilbert spaces  $\mathbb{H}_n$  converges to a hilbert  $\mathbb{H}$  if there is a family of linear maps  $\{\Phi_n : \mathbb{H} \rightarrow \mathbb{H}_n\}$  such that:*

$$\lim_{n \rightarrow +\infty} \|\Phi_n(x)\|_{\mathbb{H}_n} = \|x\|_{\mathbb{H}}, \quad x \in \mathbb{H} \quad (4.4)$$

*A sequence  $(x_n)_n$ ,  $x_n \in \mathbb{H}_n$ , converges strongly to a vector  $x \in \mathbb{H}$  if there exists a sequence  $(\tilde{x}_n)_n$  in  $\mathbb{H}$  such that  $\tilde{x}_n \rightarrow x$  in  $\mathbb{H}$  and*

$$\lim_{n \rightarrow +\infty} \overline{\lim}_{m \rightarrow +\infty} \|\Phi_m(\tilde{x}_n) - x_m\|_{\mathbb{H}_m} = 0 \quad (4.5)$$

*and  $(x_n)_n$  converge weakly to  $x$  if*

$$\lim_{n \rightarrow +\infty} \langle x_n, z_n \rangle_{\mathbb{H}_n} = \langle x, z \rangle_{\mathbb{H}} \quad (4.6)$$

*for any  $z \in \mathbb{H}$  and sequence  $(z_n)_n$ ,  $z_n \in \mathbb{H}_n$ , such that  $z_n \rightarrow z$  strongly.*

**Lemma 4.4.**

- (1) *The sequence of Hilbert spaces  $L^2(\nu_n)$  converges to  $L^2(\nu)$ , by choosing  $\Phi_n$  equal to the natural identification of equivalence classes in  $L^2(\nu_n)$  and  $L^2(\nu)$ .*



- (2)  $u_n \in L^2(\nu_n)$  converges strongly to  $u \in L^2(\nu)$  if and only if  $u_n \rightarrow u$  in  $L^2(\nu)$ .  
 (3)  $u_n \in L^2(\nu_n)$  converges weakly to  $u \in L^2(\nu)$  if and only if  $u_n \rightarrow u$  weakly in  $L^2(\nu)$ .

*Proof.* (1) We have to prove that for all  $x \in L^2(\nu)$  we have  $\|x\|_{L^2(\nu_n)} \rightarrow \|x\|_{L^2(\nu)}$  as  $n \rightarrow \infty$ . Since  $e^{-F_n}/Z_n$  converges a.s. to  $e^{-F}/Z$  and it is uniformly bounded, then the result follows by dominated convergence.

- (2) Let  $(u_n)_n$  converges strongly to  $u \in L^2(\nu)$  so there is a sequence  $(\tilde{u}_n)_n$  in  $L^2(\nu)$  tending to  $u$  in  $L^2(\nu)$  such that:

$$\lim_n \overline{\lim}_m \|\tilde{u}_n - u_m\|_{L^2(\nu)_m} = 0. \quad (4.7)$$

Then we have:

$$\overline{\lim}_m \|u - u_m\|_{L^2(\nu)} \leq \lim_n \|u - \tilde{u}_n\|_{L^2(\nu)} + \frac{C}{c} \lim_n \overline{\lim}_m \|u_m - \tilde{u}_n\|_{L^2(\nu)_m} = 0,$$

so that  $u_n \rightarrow u$  in  $L^2(\nu)$ . Conversely, if  $u_n \rightarrow u$  in  $L^2(\nu)$  then we can consider  $\tilde{u}_n = u$  for all  $n \in \mathbb{N}$  and (4.7) holds.

- (3) Let  $u_n \in L^2(\nu_n)$  be a sequence which converges weakly to  $u \in L^2(\nu)$ , i.e. for all  $v \in L^2(\nu)$  and any sequence  $v_n \in L^2(\nu_n)$  strongly convergent to  $v$

$$\langle u_n, v_n \rangle_{L^2(\nu_n)} \rightarrow \langle u, v \rangle_{L^2(\nu)}, \quad n \rightarrow +\infty.$$

Let  $v_n := v \cdot \rho \cdot \rho_n^{-1}$ , then by the dominated convergence theorem  $\|v_n - v\|_{L^2(\nu)} \rightarrow 0$  and by the previous point  $v_n \in L^2(\nu_n)$  converges strongly to  $v$ . So we have

$$\langle u_n, v \rangle_{L^2(\nu)} = \langle u_n, v_n \rangle_{L^2(\nu_n)} \rightarrow \langle u, v \rangle_{L^2(\nu)}, \quad n \rightarrow +\infty.$$

Viceversa, let us suppose that for all  $v \in L^2(\nu)$  we have  $\langle u_n, v \rangle_{L^2(\nu)} \rightarrow \langle u, v \rangle_{L^2(\nu)}$  and let us consider any sequence  $v_n \in L^2(\nu_n)$  strongly convergent to  $v$ . Setting  $w_n := v_n \cdot \rho_n \cdot \rho^{-1}$ , by dominated convergence  $\|w_n - v\|_{L^2(\nu)} \rightarrow 0$  and therefore  $\langle u_n, v_n \rangle_{L^2(\nu_n)} = \langle u_n, w_n \rangle_{L^2(\nu)} \rightarrow \langle u, v \rangle_{L^2(\nu)}$  and the proof is finished.  $\square$

**4.2. Convergence of Dirichlet Forms.** Now we can give the definition of Mosco-convergence of Dirichlet forms. This concept is useful for our purposes, since it was proved in [17] to imply the convergence in a strong sense of the associated resolvents and semigroups.

**Definition 4.5.** If  $\mathcal{E}^n$  is a quadratic form on  $\mathbb{H}_n$ , then  $\mathcal{E}^n$  Mosco-converges to the quadratic form  $\mathcal{E}$  on  $\mathbb{H}$  if the two following conditions hold:

*Mosco I.* For any sequence  $x_n \in \mathbb{H}_n$ , converging weakly to  $x \in \mathbb{H}$ ,

$$\mathcal{E}(x, x) \leq \varliminf_{n \rightarrow +\infty} \mathcal{E}^n(x_n, x_n). \quad (4.8)$$

*Mosco II.* For any  $x \in \mathbb{H}$ , there is a sequence  $x_n \in \mathbb{H}_n$  converging strongly to  $x \in \mathbb{H}$  such that

$$\mathcal{E}(x, x) = \lim_{n \rightarrow +\infty} \mathcal{E}^n(x_n, x_n). \quad (4.9)$$

We say that a sequence of bounded operators  $(B_n)_n$  on  $\mathbb{H}_n$ , converges strongly to an operator  $B$  on  $\mathbb{H}$ , if  $\mathbb{H}_n \ni B_n u_n \rightarrow Bu \in \mathbb{H}$  strongly for all sequence  $u_n \in \mathbb{H}_n$  converging strongly to  $u \in \mathbb{H}$ . Then Kuwae and Shioya have proved in [17] the following equivalence between Mosco convergence and strong convergence of the associated resolvent operators.

**Theorem 4.6** (Kuwae and Shioya [17]). *The Mosco convergence is equivalent to the strong convergence of the associated resolvents.*

### 4.3. Mosco convergence.

**Proposition 4.7.** *The Dirichlet form  $\mathcal{E}^n$  on  $L^2(\nu_n)$  Mosco-converges to  $\mathcal{E}$  on  $L^2(\nu)$ .*

*Proof.* The proof of the condition Mosco II is trivial in our case; indeed, for all  $x \in D(\mathcal{E})$ , we set  $x_n := x \in D(\mathcal{E}^n)$  for all  $n \in \mathbb{N}$ ; by dominated convergence  $\mathcal{E}(x, x) = \lim_n \mathcal{E}^n(x, x)$ . If  $x \notin D(\mathcal{E})$ , then again  $x_n := x \notin D(\mathcal{E}^n)$  satisfies  $\mathcal{E}(x, x) = \lim_n \mathcal{E}^n(x, x) = +\infty$ .

Let us prove now condition Mosco I. We first assume that  $u \in \mathcal{D}(\mathcal{E})$ . By the integration by parts formula (3.3) we have for any  $v \in \mathcal{K} = \text{Exp}_A(H)$

$$2\mathcal{E}(u, v) = - \int_H u \cdot \text{Tr}(D^2 v) d\nu + \int_H u \left( \langle \cdot, A \nabla v \rangle_H - \int_{\mathbb{R} \times [0,1]} f(da) \nabla_r v \ell^a(dr) \right) d\nu.$$

Let  $u_n \in L^2(\nu_n)$  a sequence converging weakly to  $u$ , then we know from Theorem 4.4 that  $u_n \rightarrow u$  weakly in  $L^2(\nu)$ . By the compactness of the embedding  $D(\mathcal{E}^0) \hookrightarrow L^2(\mu)$  proved in Proposition 2.3,  $u_n \rightarrow u$  strongly in  $L^2(\nu)$ . By linearity it is enough to consider  $v(x) = \exp(i\langle h, x \rangle_H)$ ,  $h \in D(A)$ ,  $x \in H$ . Notice that  $\nabla v = i v h$ . Then we can write

$$\int_{\mathbb{R} \times [0,1]} f(da) \nabla_r v(\beta) \ell^a(dr) = i v(\beta) \int_{\mathbb{R} \times [0,1]} f(da) h_r \ell^a(dr).$$

Moreover by the occupation times formula

$$\langle \nabla v(\beta), f'_n(\beta) \rangle_H = i v(\beta) \int_0^1 h_r f'_n(\beta_r) dr = i v(\beta) \int_{\mathbb{R} \times [0,1]} h_r \ell^a(dr) f'_n(a) da.$$

Since  $f'_n(a) da \rightharpoonup f(da)$ , by dominated convergence we obtain

$$2\mathcal{E}(u, v) = \lim_{n \rightarrow \infty} \left( - \int_H u^n \cdot \text{Tr}(D^2 v) d\nu^n + \int_H u^n (\langle x, A \nabla v \rangle_H + \langle \nabla v, f'_n \rangle) d\nu^n \right).$$

We can suppose that each  $u^n$  is in  $\mathcal{D}(\mathcal{E}^n)$  (else  $\mathcal{E}^n(u^n, u^n) = +\infty$ ) so we have for any  $v \in \mathcal{K} \setminus \{0\}$

$$\lim_{n \rightarrow +\infty} \left( \mathcal{E}^n(u^n, u^n) \right)^{1/2} \geq \lim_{n \rightarrow +\infty} \frac{\mathcal{E}^n(u^n, v)}{\sqrt{\mathcal{E}^n(v, v)}} = \frac{\mathcal{E}(u, v)}{\sqrt{\mathcal{E}(v, v)}}$$

and by considering the sup over  $v$  we obtain the desired result.

Suppose now that  $u \notin \mathcal{D}(\mathcal{E})$  and let  $L^2(\nu_n) \ni u^n \rightarrow u \in L^2(\nu)$  weakly, then we know from Theorem 4.4 that  $u_n \rightarrow u$  weakly in  $L^2(\nu)$ . By the compactness of the embedding  $D(\mathcal{E}^0) \hookrightarrow L^2(\mu)$  proved in Proposition 2.3,  $u_n \rightarrow u$  strongly in  $L^2(\nu)$ .

If  $\liminf_{n \rightarrow \infty} \mathcal{E}^n(u^n, u^n) < +\infty$ , then we also have  $\liminf_{n \rightarrow \infty} \mathcal{E}(u^n, u^n) < +\infty$ . But since  $\mathcal{E}$  is lower semi-continuous in  $L^2(\nu)$ , then  $\mathcal{E}(u, u) < +\infty$ , which is absurd since we assumed that  $u \notin \mathcal{D}(\mathcal{E})$ .  $\square$

**4.4. Convergence of stationary solutions.** We denote by  $\mathbb{P}_{\nu_n}^n$  the law of the stationary solution of (4.1) and by  $\mathbb{P}_\nu$  the law of the Markov process associated with  $\mathcal{E}$  and started with law  $\nu$ . We have the following convergence result

**Proposition 4.8.** *The sequence  $\mathbb{P}_{\nu_n}^n$  converges weakly to  $\mathbb{P}_\nu$  in  $C([0, T] \times [0, 1])$ .*

*Proof.* Let us first prove convergence of finite-dimensional distributions, i.e.

$$\lim_{n \rightarrow +\infty} \mathbb{E}_{\nu_n}^n(f(X_{t_1}, \dots, X_{t_m})) = \mathbb{E}_\nu(f(X_{t_1}, \dots, X_{t_m})),$$

for all  $f \in C((C([0, 1])^m)$ . The Mosco convergence of the Dirichlet forms  $\mathcal{E}^n$  provides the strong convergence of the semi-group and, by the Markov property, the convergence of the finite dimensional laws. Indeed let  $f$  be in  $C((C([0, 1])^m)$  of the form  $f(x_1, \dots, x_m) = f_1(x_1) \cdot \dots \cdot f_m(x_m)$  then

$$\begin{aligned} & P_{t_1}^n(f_1 \cdot P_{t_2-t_1}^n(f_2 \cdot \dots (f_{m-1} P_{t_m-t_{m-1}}^n f_m) \dots)) \\ & \rightarrow P_{t_1}(f_1 \cdot P_{t_2-t_1}(f_2 \cdot \dots (f_{m-1} P_{t_m-t_{m-1}} f_m) \dots)), \quad \text{strongly.} \end{aligned}$$

Then by the Markov property

$$\begin{aligned} \mathbb{E}_{\nu_n}^n(f(X_{t_1}, \dots, X_{t_m})) &= \langle 1, P_{t_1}^n(f_1 \cdot P_{t_2-t_1}^n(f_2 \cdot \dots (f_{m-1} P_{t_m-t_{m-1}}^n f_m) \dots)) \rangle_{H_n} \\ &\rightarrow \langle 1, P_{t_1}(f_1 \cdot P_{t_2-t_1}(f_2 \cdot \dots (f_{m-1} P_{t_m-t_{m-1}} f_m) \dots)) \rangle_H = \mathbb{E}_\nu(f(X_{t_1}, \dots, X_{t_m})). \end{aligned}$$

We need now to prove tightness in  $C([0, T] \times [0, 1])$ . We first recall a result of [11, Th. 7.2 ch 3]. Let  $(P, d)$  be a Polish space, and let  $(X_\alpha)_\alpha$  be a family of processes with sample paths in  $C([0, T]; P)$ . Then the laws of  $(X_\alpha)_\alpha$  are relatively compact if and only if the following two conditions hold:

- (1) For every  $\eta > 0$  and rational  $t \in [0, T]$ , there is a compact set  $\Gamma_\eta^t \subset P$  such that:

$$\inf_\alpha \mathbb{P}(X_\alpha \in \Gamma_\eta^t) \geq 1 - \eta \quad (4.10)$$

- (2) For every  $\eta, \epsilon > 0$  and  $T > 0$ , there is  $\delta > 0$  such that

$$\sup_\alpha \mathbb{P}(w(X_\alpha, \delta, T) \geq \epsilon) \leq \eta \quad (4.11)$$

where  $w(\omega, \delta, T) := \sup\{d(\omega(r), \omega(s)) : r, s \in [0, T], |r - s| \leq \delta\}$  is the modulus of continuity in  $C([0, T]; P)$ .

We consider now, as Polish space  $(P, d)$ , the Banach space  $C^\theta([0, 1])$ . Since  $\mathbb{P}_{\nu_n}^n$  is stationary, (4.10) is reduced to a condition on  $\nu_n$ . In fact we have

$$\left( \int_H \|x\|_{W^{\eta,p}(0,1)}^p d\nu_n \right)^{\frac{1}{p}} \leq \left( \frac{C}{c} \int_H \|x\|_{W^{\eta,p}(0,1)}^p d\mu \right)^{\frac{1}{p}}.$$

Now, since the Brownian bridge  $(\beta_r)_{r \in [0,1]}$  is a Gaussian process with covariance function  $r \wedge s - rs$ , then

$$\begin{aligned} \mathbb{E} \left( \|\beta\|_{W^{\eta,p}(0,1)}^p \right) &\leq \mathbb{E} \left( \|\beta\|_p^p + \int_0^1 \int_0^1 \frac{|\beta_s - \beta_t|^p}{|s - t|^{p\eta+1}} dt ds \right) \\ &\leq C_p \left( 1 + \int_0^1 \int_0^1 |s - t|^{p(\frac{1}{2}-\eta)-1} dt ds \right) < +\infty. \end{aligned}$$

For any  $\eta < 1/2$ ,  $\theta < \eta$  and  $p > 1/(\eta - \theta)$  we have by the Sobolev embedding Theorem that  $W^{\eta,p}(0,1) \subset C^\theta([0,1])$  with continuous embedding, so that  $\sup_n \int_H \|x\|_{C^\theta([0,1])}^p d\nu_n < \infty$ . By Lemma 6.1 below we obtain existence of a constant  $K$  independent of  $n$  such that

$$\mathbb{E}_{\nu_n}^n \left[ \|X_t - X_s\|_{C^\theta([0,1])}^p \right] \leq K |t - s|^\xi, \quad \forall n \geq 1, t, s \in [0, T].$$

By Kolmogorov's criterion, see [20, Thm. I.2.1], we obtain that a.s.  $w(X^n, \delta, T) \leq C \delta^{\frac{1-\xi}{2p}}$ , with  $C \in L^p$ . Therefore by the Markov inequality, if  $\epsilon > 0$

$$\mathbb{P}(w(X^n, \delta, T) \geq \epsilon) \leq \mathbb{E}[C^p] \delta^{\frac{1-\xi}{2}} \epsilon^{-p},$$

and (4.11) follows for  $\delta$  small enough.  $\square$

## 5. CONVERGENCE OF FINITE DIMENSIONAL APPROXIMATIONS

From now on we turn our attention to another problem: convergence in law of finite dimensional approximations of equation (1.6). We want to project, in a sense to be made precise, (1.6) onto an equation in a finite dimensional subspace of  $H := L^2(0,1)$ . To be more precise, we consider the space  $H_n$  of functions in  $L^2(0,1)$  which are constant on each interval  $[(i-1)2^{-n}, i2^{-n}[$ ,  $i = 1, \dots, 2^n$  and we endow  $H_n$  with the scalar product inherited from  $H$ .

Notice that  $H_n$  is a linear closed subspace of  $L^2(0,1)$ , so that there exists a unique orthogonal projector  $P_n : L^2(0,1) \mapsto H_n$ , given explicitly by

$$P_n x := 2^n \sum_{i=0}^{2^n-1} \mathbb{1}_{[i2^{-n}, (i+1)2^{-n}[} \langle \mathbb{1}_{[i2^{-n}, (i+1)2^{-n}[}, x \rangle. \quad (5.1)$$

We call  $\mu_n$  the law of  $P_n \beta$ ; then  $\mu_n$  is a Gaussian law on  $H$  with zero mean and non-degenerate covariance operator  $P_n Q P_n$ , where  $Q$  is the covariance operator of  $\mu$ , which has been studied in detail in section 2.2. In what follows we write

$$P_n Q P_n = (-2A_n)^{-1}, \quad A_n : H_n \mapsto H_n.$$

We also define  $\pi_n$  as

$$\pi_n(dx) = \frac{1}{Z_n} \exp(-F_n(x)) \mu_n(dx) = \frac{1}{Z_n} \exp\left(-\frac{1}{2^n} \sum_{i=0}^{2^n-1} f(x(i))\right) \mu_n(dx). \quad (5.2)$$

where  $Z_n := \mu_n(\exp(-F_n))$  is a normalization constant.

Then, a natural approximation of  $\mathcal{E}$  defined on  $H_n$  is given by the following symmetric bilinear non-negative form

$$\Lambda_n(u, v) := \frac{1}{2} \int \langle \nabla u, \nabla v \rangle_{H_n} d\pi_n, \quad u, v \in C^1(H_n) \quad (5.3)$$

with reference measure  $\pi_n$ . Then we have

$$\Lambda^n(u, v) = \frac{1}{2} \int \langle \nabla(u \circ P_n), \nabla(v \circ P_n) \rangle_H \frac{1}{Z_n} \exp(-F_n \circ P_n) d\mu, \quad u, v \in C^1(H_n). \quad (5.4)$$

We write

$$f(y) = f_0(y) + \sum_{j=1}^k \alpha_j \mathbb{1}_{(y \leq y_j)}, \quad y \in \mathbb{R} \quad (5.5)$$

where  $f_0$  is smooth and bounded and  $\alpha_j, y_j \in \mathbb{R}$ . Clearly,  $f$  has a jump in each  $y_j$  of respective size  $\alpha_j$ . We have the following integration by parts formula

$$\begin{aligned} \int \partial_h \varphi d\pi_n &= - \int \varphi \langle x, A_n h \rangle \pi_n(dx) + \int \varphi(x) 2^{-n} \sum_{i=0}^{2^n-1} h_i f'_0(x(i)) \pi_n(dx) \\ &\quad - \int \varphi(x) \sum_{i=0}^{2^n-1} h_i \sum_j 2 \frac{1 - e^{-\alpha_j 2^{-n}}}{1 + e^{-\alpha_j 2^{-n}}} \pi_n(dx; x(i) = y_j), \end{aligned} \quad (5.6)$$

where we use the notation

$$\pi_n(A; x(i) = y_j) := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \pi_n(A \cap \{|x(i) - y_j| \leq \varepsilon\}).$$

This suggests that the associated dynamic solves the stochastic differential equation

$$dX^i = \frac{1}{2} ((A_n X)^i - f'_0(X^i)) dt + \sum_j \frac{1 - e^{-\alpha_j 2^{-n}}}{1 + e^{-\alpha_j 2^{-n}}} d\ell_t^{i, y_j} + dw_t^i \quad (5.7)$$

where  $(\ell_t^{i, a}, t \geq 0)$  is the local time of  $(X^i(t), t \geq 0)$  at  $a$ . Then  $(X_t^i)_i$  is a vector of interacting skew Brownian motions.

**5.1. Skew Brownian motion.** Let  $(X_t)_{t \geq 0}$  be the skew Brownian motion defined in (1.1) with  $|\beta| < 1$ . Then

**Lemma 5.1.** *The process  $(X_t)_{t \geq 0}$  is associated with the Dirichlet form*

$$D(u) := \frac{1}{2} \int_{\mathbb{R}} (\dot{u})^2 \exp(-\alpha \mathbb{1}_{]-\infty, 0]}) dx$$

in  $L^2(\exp(-\alpha \mathbb{1}_{]-\infty, 0]}) dx)$ , where  $\alpha \in \mathbb{R}$  is defined by  $\frac{1-e^{-\alpha}}{1+e^{-\alpha}} = \beta$ .

*Proof.* The form  $(D, C_b^1(\mathbb{R}))$  is closable in  $L^2(\exp(-\alpha \mathbb{1}_{]-\infty, 0]}) dx)$  since it is equivalent to the standard Dirichlet forms associated with the Brownian motion. By the same argument, the closure of  $(D, C_b^1(\mathbb{R}))$  is regular and therefore there exists an

associated Hunt process  $(X_t)_{t \geq 0}$ . We want now to prove that this process is a weak solution of (1.1). The following integration by parts formula

$$\begin{aligned} \int \varphi' \exp(-\alpha \mathbb{1}_{]-\infty, 0]}) dx &= -(1 - e^{-\alpha}) \varphi(0) \\ &= 2 \frac{1 - e^{-\alpha}}{1 + e^{-\alpha}} \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \varphi \exp(-\alpha \mathbb{1}_{]-\infty, 0]}) dx, \end{aligned}$$

together with the Fukushima decomposition, shows that  $X_t$  is a semimartingale and that it satisfies (1.1) for quasi-every initial point  $X_0 = x$ , i.e. for all  $x$  outside a set  $N$  of null capacity. However, we can in fact choose  $N = \emptyset$  by noting that the transition semigroup of the skew Brownian motion with  $-1 \leq \beta \leq 1$  has an explicit Markov transition density with respect to the Lebesgue measure (see III.1.16, VII.1.23, XII.2.16 in [20]). Therefore  $X$  satisfies the *absolute continuity assumption* and we can use [12, Theorem 4.1.2 and formula (4.2.9)].  $\square$

**Theorem 5.2.** *The form  $\Lambda_n$ , defined in (5.3), is a regular Dirichlet form in  $L^2(\pi_n)$ , and the associated Markov process is a weak solution of (5.7). Moreover such solution is unique in law.*

*Proof.* As in the proof of Lemma 5.1,  $\Lambda_n$  is a regular Dirichlet form with the strong local property because it is equivalent to the Dirichlet form of a finite dimensional Ornstein-Uhlenbeck process. So by [12] there is a continuous Hunt process associated to  $\Lambda_n$ .

By the integration by parts formula (5.6) and the Fukushima decomposition, the Hunt process associated with  $\Lambda_n$  has the following property: the process  $(\langle h, X_t \rangle)_{t \geq 0}$  is a semi-martingale

$$\langle h, X_t^n \rangle - \langle h, X_0^n \rangle = M_t^h + N_t^h \quad (5.8)$$

and the Revuz measure of the bounded-variation CAF  $N^h$  is

$$\Sigma^h(dx) = \frac{1}{2} \langle A_n x - f'_0(x), h \rangle \pi_n(dx) + \sum_{i=0}^{2^n-1} h_i \sum_j \frac{1 - e^{-\alpha_j 2^{-n}}}{1 + e^{-\alpha_j 2^{-n}}} \pi_n(dx; x(i) = y_j). \quad (5.9)$$

Because of the structure of  $\Sigma^h$ , the process  $N^h$  can be written as

$$N_t^h = \int_0^t \frac{1}{2} \langle A_n X_s - f'_0(X_s), h \rangle ds + \sum_{i=0}^{2^n-1} h_i \sum_j \frac{1 - e^{-\alpha_j 2^{-n}}}{1 + e^{-\alpha_j 2^{-n}}} \ell_t^{i, y_j},$$

where  $\ell_t^{i, y_j}$  is adapted to the natural filtration of  $(X_t, t \geq 0)$ . We want now to show that in fact  $\ell_t^{i, y_j}$  is adapted to the natural filtration of  $(X_t^i, t \geq 0)$ . Since  $X_t^i$  is a semimartingale, by Tanaka's formula

$$|X_t^i - y_j| = |X_0^i - y_j| + \int_0^t \text{sign}(X_s^i - y_j) dX_s^i + L_t^{y_j}(X^i) \quad (5.10)$$

where  $L^{y_j}(X^i)$  is the local time of  $X_t^i$  at  $y_j$ . Since  $|\langle e_i, \cdot \rangle - y_j| \in \mathcal{E}_n^f$ , then  $L^{y_j}(X^i)$  is an additive functional of  $X$ . Now we can compute the Revuz measure of  $L^{y_j}(X^i)$ ,

using theorem 5.4.2 of [12]. With an integration by parts formula we see that for all  $\varphi$  smooth enough:

$$\begin{aligned}\mathcal{E}_n^f(|\langle e_i, \cdot \rangle - y_j|, \varphi) &= \frac{1}{2} \int \text{sign}(x_i - y_j) \partial_i \varphi(x) d\pi_n \\ &= -\frac{1}{2} \int \text{sign}(x_i - y_j) ((A_n x)^i - f'_0(x_i)) \varphi(x) d\pi_n - \int \varphi(x) \pi_n(dx; x(i) = y_j).\end{aligned}$$

By comparison with (5.10), we see that  $\pi_n(dx; x(i) = y_j)$  is the Revuz measure of  $t \mapsto L_t^{y_j}(X^i)$  and therefore by (5.9) the processes  $(L_t^{y_j}(X^i), t \geq 0)$  and  $(\ell_t^{i, y_j}, t \geq 0)$  are equal up to a multiplicative constant.

We want now to prove uniqueness in law for (5.7). We define the exponential martingale

$$M_t := \exp \left( - \int_0^t \frac{1}{2} \langle A_n X_s - f'_0(X_s), dw_s \rangle - \frac{1}{8} \int_0^t \|A_n X_s - f'_0(X_s)\|^2 ds \right).$$

Then under the probability measure  $M_T \cdot \mathbb{P}_x$ , by the Girsanov theorem the canonical process is a solution in law of

$$dX^i = \sum_j \frac{1 - e^{-\alpha_j 2^{-n}}}{1 + e^{-\alpha_j 2^{-n}}} d\ell_t^{i, y_j} + d\hat{w}_t^i, \quad t \in [0, T],$$

where the Brownian motions  $(\hat{w}_t^i, t \geq 0)_i$  are independent; therefore we have reduced to an independent vector of skew-Brownian motions and uniqueness in law holds for such processes by the pathwise uniqueness proved by Harrison and Shepp in [15].

Moreover, by the property recalled in the proof of Lemma 5.1, the transition semigroup of the skew-Brownian motion satisfies the absolute continuity condition and therefore all the above statements are true for all initial conditions.  $\square$

## 5.2. Convergence of the Hilbert spaces.

**Proposition 5.3.** *The sequence of Hilbert spaces  $(L^2(\pi_n))_n$  converges to  $L^2(\nu)$  in the sense of Definition 4.3.*

*Proof.* According to Definition 4.3, we have first to define a map  $\Phi_n : L^2(\nu) \mapsto L^2(\pi_n)$ . We consider now the Borel  $\sigma$ -field  $\mathcal{B}$  on  $L^2(0, 1)$ , completed with all  $\mu$ -null sets (we use the same notation for the completed  $\sigma$ -field).

Setting  $\bar{\beta} := P_n \beta$ , let us introduce the filtration  $\mathcal{F}_n := \sigma(\bar{\beta}_{i2^{-n}}, i = 1, \dots, 2^n)$  and the linear map  $\Phi_n : L^2(\mu) \mapsto L^2(\mu_n)$  defined as follows:  $\Phi_n(\varphi) = \varphi_n$ , where

$$\varphi_n(\bar{\beta}_{i2^{-n}}, i = 1, \dots, 2^{-n}) = \mathbb{E}(\varphi(\beta) | \mathcal{F}_n).$$

Then  $\varphi_n$  is well defined  $\mu_n$ -a.e. For any  $\varphi \in L^2(\mu)$  the sequence  $(\varphi_n)_n$  is a martingale bounded in  $L^2(\mu)$ , therefore converging a.s. and in  $L^2(\mu)$ . Now, since  $L^2(\mu) \equiv L^2(\nu)$  and  $L^2(\mu_n) \equiv L^2(\pi_n)$  with equivalence of norms (uniformly in  $n$ ), then the map  $\Phi_n$  is still well defined and  $\sup_n \|\varphi_n\|_{L^2(\pi_n)} < +\infty$  for all  $\varphi \in L^2(\nu)$ . We have to prove that  $\|\varphi_n\|_{L^2(\pi_n)} \rightarrow \|\varphi\|_{L^2(\nu)}$  as  $n \rightarrow +\infty$ .

We first prove that  $F_n(\bar{\beta}^n)$  converges a.s. to  $F(\beta)$ , where  $\beta^n := \bar{\beta}_{\lfloor r2^n \rfloor}$ ,  $r \in [0, 1]$ . We have that

$$F_n(\beta^n) = 2^n \sum_{i=1}^{2^n-1} f_n(\beta_{i2^{-n}}) = \int_0^1 f_n(\beta_{\lfloor r2^n \rfloor}) dr.$$

Now by dominated convergence it is enough to prove that a.s.  $f_n(\beta_r^n) \xrightarrow{n \rightarrow +\infty} f(\beta_r)$  for a.e.  $r \in [0, 1]$ . By (5.5),  $f$  is continuous outside the finite set  $\Delta_f = \{y_j\}$ . Moreover  $(f_n)_n$  is uniformly bounded and  $f_n \xrightarrow{n \rightarrow +\infty} f$  as  $n \rightarrow +\infty$  locally uniformly in  $\mathbb{R} \setminus \Delta_f$ . For all  $a \in \mathbb{R}$ , a.s.  $\{r \in [0, 1] : \beta_r = a\}$  is a compact set with zero Lebesgue measure and therefore a.s.  $U := \{r \in [0, 1] : \beta_r \in \Delta_f\}$  also has zero Lebesgue measure. Therefore for all  $r \in [0, 1] \setminus U$ ,  $f_n(\beta_r^n) \xrightarrow{n \rightarrow +\infty} f(\beta_r)$  and by dominated convergence  $F_n(\beta^n)$  converges a.s. to  $F(\beta)$ . In particular, by dominated convergence  $Z_n = \mu_n(e^{-F_n}) = \mathbb{E}(e^{-F_n(\bar{\beta}^n)})$  converges to  $Z = \mathbb{E}(e^{-F(\beta)})$ .

Now, let us prove that  $\|\varphi_n\|_{L^2(\pi_n)} \rightarrow \|\varphi\|_{L^2(\nu)}$ . Since  $Z_n \xrightarrow{n \rightarrow +\infty} Z$ , we have to prove that

$$\mathbb{E} \left( \varphi_n^2(\bar{\beta}^n) e^{-F_n(\bar{\beta}^n)} \right) \xrightarrow{n \rightarrow +\infty} \mathbb{E} \left( \varphi^2(\beta) e^{-F(\beta)} \right).$$

We have shown above that  $\varphi_n(\bar{\beta}^n)$  converges to  $\varphi(\beta)$  in  $L^2$ . Therefore  $(\varphi_n^2(\bar{\beta}^n))_n$  is uniformly integrable and so is also  $(\varphi_n^2(\bar{\beta}^n) e^{-F_n(\bar{\beta}^n)})_n$ , since  $(e^{-F_n(\bar{\beta}^n)})_n$  is bounded in  $L^\infty$ . We can then conclude since a u.i. sequence converging a.s. converges in  $L^1$ .  $\square$

**5.3. Mosco convergence.** We want now to prove that  $\Lambda^n$  Mosco converges to  $\mathcal{E}$ . In [3, Thm. 3.5], Andres and von Renesse have proved that Theorem 4.6 still holds if one replaces the condition *Mosco II* with the following condition *Mosco II'*.

**Definition 5.4** (*Mosco II'*). *There is a core  $\mathcal{K} \subset \mathcal{D}(\mathcal{E})$  such that for any  $x \in \mathcal{K}$  there exists a sequence  $x_n \in \mathcal{D}(\Lambda^n)$  converging strongly to  $x$  and such that  $\mathcal{E}(x, x) = \lim_{n \rightarrow +\infty} \Lambda^n(x_n, x_n)$ .*

**Theorem 5.5** (Andres and von Renesse [3]). *The conditions Mosco I and Mosco II' are equivalent to the Mosco convergence.*

**Theorem 5.6.** *The Dirichlet form  $\Lambda_n$  Mosco-converges to  $\Lambda$  as  $n \rightarrow +\infty$ .*

**Lemma 5.7.** *Let  $u_n \in L^2(\pi_n)$  be a sequence which converges weakly to  $u \in L^2(\nu)$ , and such that  $\liminf_n \Lambda^n(u_n, u_n) < +\infty$ , then there is a subsequence of  $(u_n \circ P_n)_n$  converging to  $u$  in  $L^2(\nu)$ .*

*Proof.* By passing to a subsequence, we can suppose that  $\limsup_n \Lambda^n(u_n, u_n) < +\infty$ . By (5.4), we have that  $\mathcal{E}^0(u_n \circ P_n, u_n \circ P_n) \leq C \Lambda^n(u_n, u_n)$ , for some constant  $C > 0$ , and therefore  $\limsup_n \mathcal{E}^0(u_n \circ P_n, u_n \circ P_n) < +\infty$ . By Proposition 2.1, the inclusion  $D(\mathcal{E}^0) \subseteq L^2(\nu)$  is compact, so that we can extract a subsequence  $v_{n_k} := u_{n_k} \circ P_{n_k}$  converging in  $L^2(\nu)$ . This subsequence  $u_{n_k} \in L^2(\pi_{n_k})$  converges strongly to  $u \in L^2(\nu)$ , since  $\Phi_n(u_n \circ P_n) = u_n$ , by the definition of  $\Phi_n$  given in the proof of Proposition 5.3.  $\square$



*Proof of Theorem 5.6.* Let us consider the following regularization of  $f$ : we fix a function  $\rho : \mathbb{R} \mapsto \mathbb{R}$  such that  $\rho(x) = 1$  for all  $x \leq 0$ ,  $\rho(x) = 0$  for all  $x \geq 1$ ,  $\rho$  is monotone non-increasing and twice continuously differentiable on  $\mathbb{R}$  with  $0 \leq \rho' \leq 1$ ; then we set

$$f_n(y) = f_0(y) + \sum_{j=1}^k \alpha_j \rho(n(y - y_j) + \mathbb{1}_{(\alpha_j < 0)}), \quad y \in \mathbb{R}.$$

Notice that  $f_n \downarrow f$  pointwise as  $n \uparrow +\infty$ . Now we define the measure

$$\tilde{\pi}_n(dx) = \frac{1}{Z_n} \exp(-F_n(x)) \mu_n(dx) = \frac{1}{Z_n} \exp\left(-\frac{1}{2^n} \sum_{i=1}^{2^n-1} f_n(x(i2^{-n}))\right) \mu_n(dx);$$

note that  $\tilde{\pi}_n$  is not normalized to be a probability measure, in fact  $\tilde{\pi}_n \leq \pi_n$  since  $f_n \geq f$ . We also define the Dirichlet form

$$\tilde{\Lambda}^n(\varphi, \psi) := \frac{1}{2} \int \langle \nabla \varphi, \nabla \psi \rangle d\tilde{\pi}_n, \quad \forall \varphi, \psi \in D(\Lambda^n).$$

The form  $\tilde{\Lambda}^n$  is clearly equivalent to  $\Lambda^n$  on  $D(\Lambda^n)$ . Moreover  $\tilde{\Lambda}^n(u, u) \leq \Lambda^n(u, u)$  for all  $u \in D(\Lambda^n)$ .

Let us show first condition Mosco II'. For  $v \in \mathcal{K} := \text{Exp}_A(H)$ , we have that

$$v(w) = \sum_{m=1}^k \lambda_k \exp(i\langle w, h_m \rangle)$$

and we can suppose that  $v \neq 0$ . We set  $v_n := v|_{H_n}$ . Then it is easy to see that  $v_n$  converges strongly to  $v$ ; indeed, setting  $\tilde{v}_n := v \circ P_n$ , we have  $\Phi_m(\tilde{v}_n) = v_n$  for  $m \geq n$  by construction; therefore

$$\|\Phi_m(\tilde{v}_n) - v_m\|_{L^2(\pi_m)} = \|v_n - v_m\|_{L^2(\pi_m)} \leq C \|v \circ P_n - v \circ P_m\|_{L^2(\mu)},$$

which tends to 0 as  $m \rightarrow +\infty$  and then  $n \rightarrow +\infty$ . Moreover

$$\Lambda^n(v_n, v_n) = \frac{1}{2} \int \|P_n \nabla v\|_H^2 d\pi_n \rightarrow \mathcal{E}(v, v),$$

so that Mosco II' holds.

Let us prove now Mosco I. Let  $u_n \in L^2(\pi_n)$  be a sequence converging weakly to  $u \in L^2(\nu)$ ; we can suppose that  $u \in \mathcal{D}(\mathcal{E})$  and that  $\liminf_n \Lambda^n(u_n, u_n) < +\infty$ ; then by lemma 5.7, up to passing a subsequence, we can suppose that  $u_n \rightarrow u$  strongly.

Since  $\tilde{\Lambda}^n \leq \Lambda^n$ , we have

$$\liminf_{n \rightarrow \infty} \Lambda^n(u_n, u_n) \geq \liminf_{n \rightarrow \infty} \tilde{\Lambda}^n(u_n, u_n).$$

Now for any  $v_n \in D(\Lambda^n)$

$$\tilde{\Lambda}^n(u_n, u_n) \geq \frac{(\tilde{\Lambda}^n(u_n, v_n))^2}{\tilde{\Lambda}^n(v_n, v_n)}. \quad (5.11)$$

Suppose that  $v \neq 0$  and  $v \in \text{Exp}_A(H)$  is a linear combination of exponential functions. We set  $v_n := v|_{H_n}$ . Then arguing as above we have  $\tilde{\Lambda}^n(v_n, v_n) \rightarrow \mathcal{E}(v, v)$ .

Now we prove that  $\tilde{\Lambda}^n(u_n, v_n) \rightarrow \mathcal{E}(u, v)$ . By linearity, we can suppose that  $v = \exp(i\langle \cdot, h \rangle)$ . Integrating by parts we see that

$$2\tilde{\Lambda}^n(u_n, v_n) = -i \int u_n(x) v_n(x) \langle A_n x - f'_n(x), P_n h \rangle \pi_n(dx).$$

The claim follows if we prove that

$$\int u_n(x) v_n(x) \langle n\rho'(n(x-y)), P_n h \rangle \pi_n(dx) \rightarrow \int u(x) v(x) \langle \ell^y, h \rangle \nu(dx).$$

Note that, with the notation  $\beta^n = P_n \beta$ ,

$$\int \varphi(x) \langle n\rho'(n(x-y)), h \rangle \pi_n(dx) = \mathbb{E}(\varphi(\beta^n) \langle n\rho'(n(\beta^n - y)), h \rangle).$$

Now

$$|\langle n\rho'(n(\beta^n - y)) - n\rho'(n(\beta - y)), P_n h \rangle| \leq n \sup_{|r-s| \leq 2^{-n}} |\beta_r - \beta_s| \|h\|_\infty.$$

Moreover, if  $h$  has support in  $[\varepsilon, 1 - \varepsilon]$ , then

$$\begin{aligned} & \left| \langle n\rho'(n(\beta - y)), h_n \rangle - \int_0^1 h_n d\ell^y \right| = \left| \int h_n(r) \left( \int n\rho'(n(a-y)) (\ell^a - \ell^y)(dr) da \right) \right| \\ &= \left| \int_\varepsilon^{1-\varepsilon} h'_n(r) \left( \int n\rho'(n(a-y)) (\ell^a(r) - \ell^y(r)) da \right) dr \right| \\ &\leq \|h'\| \sup_{|a-y| \leq 1/n} \sup_{r \in [\varepsilon, 1-\varepsilon]} |\ell^a(r) - \ell^y(r)|. \end{aligned}$$

We want now to show that these quantities converge to 0 in  $L^2$  as  $n \rightarrow +\infty$ . Indeed, since  $(\beta_{1-r}, r \in [0, 1])$  has the same law as  $(\beta_r, r \in [0, 1])$ , we can write

$$\begin{aligned} & \mathbb{E} \left( \sup_{|r-s| \leq 2^{-n}} |\beta_r - \beta_s|^2 \right) \leq 2 \mathbb{E} \left( \sup_{|r-s| \leq 2^{-n}, r, s \leq \frac{3}{4}} |\beta_r - \beta_s|^2 \right) \\ &= 2 \mathbb{E} \left( \sup_{|r-s| \leq 2^{-n}, r, s \leq \frac{3}{4}} |B_r - B_s|^2 \frac{p_{1/4}(B_{3/4})}{p_1(0)} \right) \leq 4 \mathbb{E} \left( \sup_{|r-s| \leq 2^{-n}, r, s \leq \frac{3}{4}} |B_r - B_s|^2 \right) \\ &\leq C(2^{-n})^{1/2} \end{aligned}$$

by Kolmogorov's continuity criterion for the standard Brownian motion  $(B_r)_{r \geq 0}$ . For the other term, we also reduce to a known result on the local time  $(\ell_t^a)_{a \in \mathbb{R}, t \geq 0}$  of Brownian motion:

$$\begin{aligned} & \mathbb{E} \left( \sup_{|a-y| \leq 1/n} \sup_{r \in [\varepsilon, 1-\varepsilon]} |\ell^a(r) - \ell^y(r)|^2 \right) \\ &= \mathbb{E} \left( \sup_{|a-y| \leq 1/n} \sup_{r \in [\varepsilon, 1-\varepsilon]} |\ell^a(r) - \ell^y(r)|^2 \frac{p_\varepsilon(B_{1-\varepsilon})}{p_1(0)} \right) \\ &\leq \varepsilon^{-1/2} \mathbb{E} \left( \sup_{|a-y| \leq 1/n} \sup_{r \in [\varepsilon, 1-\varepsilon]} |\ell^a(r) - \ell^y(r)|^2 \right) \leq C(1/n)^{1/2}, \end{aligned}$$

see [20] p.225-226. It only remains to prove that

$$\lim_n \int u_n v_n \langle A_n x - f'_0(x), P_n h \rangle \pi_n(dx) = \int u (\langle x, Ah \rangle - \langle f'_0(x), h \rangle) \nu(dx). \quad (5.12)$$

The term containing  $f'_0(x)$  gives no difficulty; as for  $\int u_n v_n \langle \cdot, A^n P_n h \rangle d\pi_n$ , we have

$$\int u_n v_n \langle \cdot, A^n P_n h \rangle d\pi_n = \frac{1}{Z_n} \int u_n v_n \langle \cdot, A^n P_n h \rangle e^{-F_n} d\mu_n.$$

Now, notice that by an integration by part formula, we have for all  $g \in C_b^1(H)$

$$\int g \circ P_n \langle \cdot, A^n P_n h \rangle d\mu = \int g \langle \cdot, A^n P_n h \rangle d\mu_n = - \int \partial_{P_n h} g d\mu_n$$

and, again by an integration by parts formula,

$$- \lim_{n \rightarrow +\infty} \int \partial_{P_n h} g d\mu_n = - \int \partial_h g d\mu = \int g \langle \cdot, Ah \rangle d\mu.$$

Moreover

$$\int \langle \cdot, A^n P_n h \rangle^2 d\mu = \int \langle \cdot, A^n P_n h \rangle^2 d\mu_n = \|P_n h\|^2 \leq \|h\|^2.$$

Therefore, the linear functional  $L^2(\mu) \ni g \mapsto \int g \circ P_n \langle \cdot, A^n P_n h \rangle d\mu$  is uniformly bounded in  $n$  and converges on  $C_b^1(H)$ , a dense subset in  $L^2(\mu)$ . By a density argument, this sequence of functionals converges weakly in  $L^2(\mu)$ .

We recall now that  $L^2(\pi_n) \ni u_n$  converges strongly to  $u \in L^2(\nu)$ . We want to show that  $(u_n v_n e^{-F_n}) \circ P_n \rightarrow u v e^{-F}$  in  $L^2(\mu)$ . Indeed by lemma 5.7, from any subsequence of  $(u_n \circ P_n)_n$  we can extract a sub-subsequence converging to  $u$  in  $L^2(\nu)$  and  $\nu$ -almost surely. On the other hand  $(v_n e^{-F_n}) \circ P_n$  converges pointwise to  $v e^{-F}$  and  $((v_n e^{-F_n}) \circ P_n)_n$  is uniformly bounded, so we conclude with the dominated convergence theorem. Therefore, we obtain that

$$\lim_n \int u_n v_n e^{-F_n} \langle \cdot, A^n P_n h \rangle d\mu_n = \int u v e^{-F} \langle \cdot, Ah \rangle d\mu,$$

and (5.12) is proved.

Finally we prove that if  $\liminf_n \Lambda^n(u_n, u_n) < +\infty$ , then  $u \in \mathcal{D}(\mathcal{E})$ . Indeed for all  $u_n \in \mathcal{D}(\Lambda^n)$  we have  $u_n \circ P_n \in \mathcal{D}(\mathcal{E})$ , moreover  $(u_n)_n$  converges weakly to  $u$  then  $(u_n \circ P_n)_n$  converges weakly to  $u$  in  $L^2(\nu)$ ; then, as at the end of the proof of Proposition 4.7, by the compact injection of  $\mathcal{D}(\mathcal{E})$  in  $L^2(\nu)$  we have that  $u \in \mathcal{D}(\mathcal{E})$ , which ends the proof.  $\square$

**5.4. Convergence in law of stationary processes.** We denote now by  $(\mathbb{Q}_{\pi_n}^n)_n$  the law of the stationary solution of equation (5.7) started with initial law  $\pi_n$ . We want to prove a convergence result for  $(\mathbb{Q}_{\pi_n}^n)_n$  to  $\mathbb{P}_\nu$ , the stationary solution to equation (1.6). We define the space  $H^{-1}(0,1)$  as the completion of  $L^2(0,1)$  with respect to the Hilbertian norm

$$\|x\|_{H^{-1}(0,1)}^2 := \int_0^1 d\theta \langle x, \mathbb{1}_{[0,\theta]} \rangle_{L^2(0,1)}^2,$$

and the linear isometry  $J : H^{-1}(0, 1) \mapsto L^2(0, 1)$  given by the closure of

$$H^{-1}(0, 1) \subset L^2(0, 1) \ni x \mapsto Jx := \langle x, \mathbb{1}_{[0, \cdot]} \rangle_{L^2(0, 1)}.$$

**Lemma 5.8.** *The sequence  $\mathbb{Q}_{\pi_n}^n$  converges weakly to  $\mathbb{P}_\nu$  in  $C([0, T]; H^{-1}(0, 1))$ .*

*Proof.* We define  $\mathbb{S}_n := \mathbb{Q}_{\pi_n}^n \circ J^{-1}$ , i.e. the law of  $(JX_t^n)_{t \geq 0}$ , where  $X_t^n$  has law  $\mathbb{Q}_{\pi_n}^n$ . Since  $J$  maps  $L^2(0, 1)$  continuously into  $H^1(0, 1)$ , we obtain that  $\pi_n^n \circ J^{-1}$  satisfies condition (6.1) below. Therefore by Lemma 6.1 below,  $(\mathbb{S}_n)_n$  is tight in  $C([0, T] \times [0, 1])$  and therefore  $(\mathbb{Q}_{\pi_n}^n)_n$  is tight in  $C([0, T]; H^{-1}(0, 1))$ .

Let us now prove convergence of finite dimensional distributions. As in the proof of Proposition 4.8, let  $f \in C_b(H^m)$  of the form  $f(x_1, \dots, x_m) = f_1(x_1) \cdots f_m(x_m)$ . By the Markov property, it is enough to prove that

$$\begin{aligned} & P_{t_1}^n(f_1 \cdot P_{t_2-t_1}^n(f_2 \cdot \dots (f_{m-1} P_{t_m-t_{m-1}}^n f_m) \dots)) \\ & \rightarrow P_{t_1}(f_1 \cdot P_{t_2-t_1}(f_2 \cdot \dots (f_{m-1} P_{t_m-t_{m-1}} f_m) \dots)), \quad \text{strongly.} \end{aligned}$$

Arguing by recurrence, we only need to prove that, if  $L^2(\pi_n) \ni v_n \rightarrow v \in L^2(\nu)$  strongly, and  $g \in C_b(H)$ , then  $L^2(\pi_n) \ni g \cdot v_n$  converges strongly to  $g \cdot v \in L^2(\nu)$ . We have

$$\begin{aligned} & \|\Phi_m(g \cdot \tilde{v}_n) - g \cdot v_m\|_{L^2(\pi_m)} \\ & \leq \|\Phi_m(g \cdot \tilde{v}_n - g \circ P_m \cdot \tilde{v}_n)\|_{L^2(\pi_m)} + \|g \cdot (\Phi_m(\tilde{v}_n) - v_m)\|_{L^2(\pi_m)}. \end{aligned}$$

Recalling that  $\Phi_m$  is defined in terms of a conditional expectation, see the proof of Proposition 5.3, we obtain

$$\limsup_m \|\Phi_m(g \cdot \tilde{v}_n - g \circ P_m \cdot \tilde{v}_n)\|_{L^2(\pi_m)} \leq \limsup_m C \|(g - g \circ P_m) \tilde{v}_n\|_{L^2(\nu)} = 0,$$

since the conditional expectation is a contraction in  $L^2(\mu)$  and  $g \circ P_m$  converges almost surely to  $g$  if  $m \rightarrow +\infty$ . Moreover

$$\lim_n \limsup_m \|g \cdot (\Phi_m(\tilde{v}_n) - v_m)\|_{L^2(\pi_m)} \leq \|g\|_\infty \lim_n \limsup_m \|\Phi_m(\tilde{v}_n) - v_m\|_{L^2(\pi_m)} = 0$$

by assumption. Therefore  $L^2(\pi_n) \ni g \cdot v_n$  converges strongly to  $g \cdot v \in L^2(\nu)$  and we obtain the convergence in law of the finite dimensional laws.  $\square$

## 6. A PRIORI ESTIMATE

We prove in this section an estimate which has been used above to prove tightness properties in  $C([0, T] \times [0, 1])$ . We consider here a probability measure  $\gamma$  on  $H$  and Dirichlet form  $(\mathbb{D}, D(\mathbb{D}))$  in  $L^2(\gamma)$  such that  $C_b^1(H)$  is a core of  $\mathbb{D}$  and

$$\mathbb{D}(u, v) = \frac{1}{2} \int \langle \nabla u, \nabla v \rangle d\gamma, \quad \forall u, v \in C_b^1(H).$$

Let us define for  $\eta \in ]0, 1[$  and  $r \geq 1$  the norm  $\|\cdot\|_{W^{\eta, r}(0, 1)}$ , given by

$$\|x\|_{W^{\eta, r}(0, 1)}^r = \int_0^1 |x_s|^r ds + \int_0^1 \int_0^1 \frac{|x_s - x_t|^r}{|s - t|^{r\eta+1}} dt ds.$$

Then we have the following

**Lemma 6.1.** *Let  $(X_t)_{t \geq 0}$  be the stationary Markov process associated with  $\mathbb{D}$ , i.e. such that the law of  $X_0$  is  $\gamma$ . Suppose that there exist  $\eta \in ]0, 1[$ ,  $\zeta > 0$  and  $p > 1$  such that*

$$\zeta > \frac{1}{1 + \frac{2}{3}\eta}, \quad p > \max \left\{ \frac{2}{1 - \zeta}, \frac{1}{\eta - \frac{3}{2} \frac{1 - \zeta}{\zeta}} \right\},$$

and

$$\int_H \|x\|_{W^{\eta,p}(0,1)}^p \gamma(dx) = C_{\eta,p} < +\infty. \quad (6.1)$$

Then there exist  $\theta \in ]0, 1[$ ,  $\xi > 1$  and  $K > 0$ , all depending only on  $(\eta, \zeta, p)$ , such that

$$\mathbb{E} \left[ \|X_t - X_s\|_{C^\theta([0,1])}^p \right] \leq K |t - s|^\xi.$$

*Proof.* We follow the proof of Lemma 5.2 in [8]. We introduce first the space  $H^{-1}(0, 1)$ , completion of  $L^2(0, 1)$  w.r.t. the norm:

$$\|f\|_{-1}^2 := \sum_{k=1}^{\infty} k^{-2} |\langle f, e_k \rangle_{L^2(0,1)}|^2$$

where  $e_k(r) := \sqrt{2} \sin(\pi k r)$ ,  $r \in [0, 1]$ ,  $k \geq 1$ , are the eigenvectors of the second derivative with homogeneous Dirichlet boundary conditions at  $\{0, 1\}$ . Recall that  $L^2(0, 1) = H$ , in our notation. We denote by  $\kappa$  the Hilbert-Schmidt norm of the inclusion  $H \rightarrow H^{-1}(0, 1)$ , which by definition is equal in our case to

$$\kappa = \sum_{k \geq 1} k^{-2} < +\infty.$$

We claim that for all  $p > 1$  there exists  $C_p \in (0, \infty)$ , depending only on  $p$ , such that

$$\left( \mathbb{E} \left[ \|X_t - X_s\|_{H^{-1}(0,1)}^p \right] \right)^{\frac{1}{p}} \leq C_p \kappa |t - s|^{\frac{1}{2}}, \quad t, s \geq 0. \quad (6.2)$$

To prove (6.2), we fix  $T > 0$  and use the Lyons-Zheng decomposition, see e.g. [12, Th. 5.7.1], to write for  $t \in [0, T]$  and  $h \in H$ :

$$\langle h, X_t - X_0 \rangle_H = \frac{1}{2} M_t - \frac{1}{2} (N_T - N_{T-t}),$$

where  $M$ , respectively  $N$ , is a martingale w.r.t. the natural filtration of  $X$ , respectively of  $(X_{T-t}, t \in [0, T])$ . Moreover, the quadratic variations are both equal to:  $\langle M \rangle_t = \langle N \rangle_t = t \cdot \|h\|_H^2$ . By the Burkholder-Davis-Gundy inequality we can find  $c_p \in (0, \infty)$  for all  $p > 1$  such that:  $(\mathbb{E} [|\langle X_t - X_s, e_k \rangle|^p])^{\frac{1}{p}} \leq c_p |t - s|^{\frac{1}{2}}$ ,  $t, s \in [0, T]$ , and therefore

$$\begin{aligned} \left( \mathbb{E} \left[ \|X_t - X_s\|_{H^{-1}(0,1)}^p \right] \right)^{\frac{1}{p}} &\leq \sum_{k \geq 1} k^{-2} (\mathbb{E} [|\langle X_t - X_s, e_k \rangle|^p])^{\frac{1}{p}} \\ &\leq c_p \sum_{k \geq 1} k^{-2} |t - s|^{\frac{1}{2}} \|e_k\|_{L^2(0,1)}^2 \leq c_p \kappa |t - s|^{\frac{1}{2}}, \quad t, s \in [0, T], \end{aligned}$$

and (6.2) is proved. By stationarity

$$\begin{aligned} \left( \mathbb{E} \left[ \|X_t - X_s\|_{W^{\eta,p}(0,1)}^p \right] \right)^{\frac{1}{p}} &\leq \left( \mathbb{E} \left[ \|X_t\|_{W^{\eta,p}(0,1)}^p \right] \right)^{\frac{1}{p}} + \left( \mathbb{E} \left[ \|X_s\|_{W^{\eta,p}(0,1)}^p \right] \right)^{\frac{1}{p}} \\ &= 2 \left( \int_H \|x\|_{W^{\eta,p}(0,1)}^p d\gamma \right)^{\frac{1}{p}} = 2 (C_{\eta,p})^{1/p}. \end{aligned} \quad (6.3)$$

By the assumption on  $\zeta$  and  $p$  it follows that  $\alpha := \zeta\eta - (1 - \zeta) > 0$  and

$$\frac{p}{2} (1 - \zeta) > 1, \quad \frac{1}{d} := \zeta \frac{1}{p} + (1 - \zeta) \frac{1}{2} < \alpha.$$

Then by interpolation, see [1, Chapter 7],

$$\begin{aligned} \left( \mathbb{E} \left[ \|X_t - X_s\|_{W^{\alpha,d}(0,1)}^p \right] \right)^{\frac{1}{p}} &\leq \\ &\leq \left( \mathbb{E} \left[ \|X_t - X_s\|_{W^{\eta,p}(0,1)}^p \right] \right)^{\frac{\zeta}{p}} \left( \mathbb{E} \left[ \|X_t - X_s\|_{H^{-1}(0,1)}^p \right] \right)^{\frac{1-\zeta}{p}}. \end{aligned}$$

Since  $\alpha d > 1$ , there exists  $\theta > 0$  such that  $(\alpha - \theta)d > 1$ . By the Sobolev embedding,  $W^{\alpha,d}(0,1) \subset C^\theta([0,1])$  with continuous embedding. Then we find that

$$\mathbb{E} \left[ \|X_t - X_s\|_{C^\theta([0,1])}^p \right] \leq K |t - s|^\xi$$

with  $\xi := \frac{p}{2} (1 - \zeta) > 1$  and  $K$  a constant depending only on  $(\eta, \zeta, p)$ .  $\square$

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